

On the eigenvalues for slowly varying perturbations of a periodic Schrödinger operator

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Abstract

In this paper, I consider one-dimensional periodic Schrödinger operators perturbed by a slowly decaying potential. In the adiabatic limit, I give an asymptotic expansion of the eigenvalues in the gaps of the periodic operator. When one slides the perturbation along the periodic potential, these eigenvalues oscillate. I compute the exponentially small amplitude of the oscillations.

Keywords: eigenvalues, complex WKB method, scattering, adiabatic perturbations

1 Introduction

In this paper, we study the spectrum of one-dimensional perturbed periodic Schrödinger operators. Precisely, we consider the Schrödinger operator defined on $L^2(\mathbb{R})$ by:

$$H_{\varphi, \varepsilon} = -\frac{d^2}{dx^2} + [V(x) + W(\varepsilon x + \varphi)], \quad (1.1)$$

where $\varepsilon > 0$ is a small positive parameter, φ is a real parameter, and V is a real valued 1-periodic function. We also assume that V is L^2_{loc} and that W is a fast-decaying function.

The operator $H_{\varphi, \varepsilon}$ can be regarded as an adiabatic perturbation of the periodic operator H_0 :

$$H_0 = -\Delta + V. \quad (1.2)$$

The spectrum of the periodic operator H_0 is absolutely continuous and consists of intervals of the real axis called the spectral bands, separated by the gaps.

If the perturbation W is relatively compact with respect to H_0 , there are in the gaps of H_0 some eigenvalues [28, 21]. We intend to locate these eigenvalues, called impurity levels.

The equation

$$H_{\varphi, \varepsilon} \psi = E \psi \quad (1.3)$$

depends on two parameters ε et φ . We study the operator $H_{\varphi, \varepsilon}$ in the adiabatic limit, i.e as $\varepsilon \rightarrow 0$. The periodicity of V implies that the eigenvalues of $H_{\varphi, \varepsilon}$ are ε -periodic in φ . We shall shift φ in the complex plane and we shall assume that W is analytic in a strip of the complex plane.

If $V = 0$, there are many results. The case when W is a well has been studied; in the interval $\inf_{\mathbb{R}} W, 0[$, there is a quantified sequence of eigenvalues [5]. We shall give an analogous description of the eigenvalues of $H_{\varphi, \varepsilon}$ in an interval J out of the spectrum of H_0 . Precisely, when W and J satisfy some additional conditions described in sections 2.2.1, 2.2.3 et 2.4, we show that the eigenvalues of $H_{\varphi, \varepsilon}$ oscillate around some quantized energies. The quantization is given by a Bohr-Sommerfeld quantization rule; the amplitude of oscillation is exponentially small and is determined by a tunneling coefficient.

1.1 Physical motivation

The operator $H_{\varphi, \varepsilon}$ is an important model of solid state physics. The function ψ is the wave function of an electron in a crystal with impurities. V represents the potential of the perfect crystal; as such it is periodic. The potential W is the perturbation created by impurities. In the semiconductors, this perturbation is slow-varying [29]. It is natural to consider the semi-classical limit.

1.2 Perturbation of periodic operators

In \mathbb{R}^d , the spectral theory of the perturbations of a periodic operator

$$H_P = H_0 + P \quad (1.4)$$

has motivated numerous studies with different view points.

The characterization of the existence of eigenvalues is not easy: particularly, in any dimension, [13] deals with the existence of embedded eigenvalues in the bands. On the real axis, the situation is simpler. When the perturbation is integrable, the eigenvalues are necessarily in the adherence of the gaps ([22, 12]).

To count the eigenvalues in the gaps, many results have been obtained thanks to trace formulas. In the large coupling constant limit, i.e when $P = \lambda U$, with $\lambda \rightarrow +\infty$, [23, 1, 24] have studied $\lim_{\lambda \rightarrow +\infty} \text{tr}(P_{[E, E']}^{(\lambda)})$,

where $P_{[E, E']}^{(\lambda)} = 1_{[E, E']} H_\lambda$ (spectral projector of H_λ on an interval $[E, E']$ of a gap of H_0). In the semi-classical case, [4] has given, under assumptions close to mine, an asymptotic expansion of $\text{tr}[f(H_{\varphi, \varepsilon})]$, for $f \in C_0^\infty(\mathbb{R})$ and $\text{Supp } f$ in a gap of H_0 . These formulas are valid in any dimension but are less accurate. For example, in the expansion obtained in [4], the accuracy depends on the successive derivatives of the function f ; the formula does not give an exponentially precise localization of the eigenvalues.

In the one-dimensional case, the scattering theory, well-known in the case $V = 0$, has been developed in [10, 17] for the periodic case. Precisely, we construct some particular solutions of equation (1.4), which tend to zero as x tends to infinity. We call these functions recessive functions. The eigenvalues of equation (1.3) are given by a relation of linear dependence between these solutions.

1.3 Main steps of the study

We give here the main ideas of the paper. An important difficulty is the dependence of the equation on the parameters ε and φ ; particularly, one has to decouple the “fast” variable x and the “slow” variable εx . The new idea developed in [6, 8] is the following : we construct some particular solutions of (1.3), satisfying an additional relation called the consistency condition:

$$f(x + 1, \varphi, E, \varepsilon) = f(x, \varphi + \varepsilon, E, \varepsilon). \quad (1.5)$$

This condition relates their behavior in x and their behavior in φ .

To find a recessive solution of (1.3), it suffices to construct a solution of (1.3) which satisfies (1.5) and which tends to 0 as $|\text{Re } \varphi|$ tends to $+\infty$. First, we build on the horizontal half-strip $\{\varphi \in \mathbb{C} ; \varphi \in]-\infty, -A] + i[-Y, Y]\}$ a solution h_-^g of equation (1.3) which is consistent and which tends to 0 as $\text{Re } \varphi$ tends to $-\infty$. Similarly, we construct h_+^d for $\{\varphi \in \mathbb{C} ; \varphi \in [A, +\infty[+ i[-Y, Y]\}$ (Theorem 2). These functions are recessive for the variable x . The characterization of the eigenvalues is given by the relation of linear dependence between h_-^g and h_+^d :

$$w(h_-^g, h_+^d) = 0.$$

In the above-mentioned equation, w represents the Wronskian whose definition is recalled in (3.4).

It remains to compute $w(h_-^g, h_+^d)$. To do that, we use the complex WKB method developed by A. Fedotov and F. Klopp. This method consists in describing some complex domains, called canonical domains, on which we construct some functions satisfying (1.5) and having a particular asymptotic behavior:

$$f_\pm(x, \varphi, E, \varepsilon) = e^{\pm \frac{i}{\varepsilon} \int^\varphi \kappa} (\psi_\pm(x, \varphi, E) + o(1)), \quad \varepsilon \rightarrow 0. \quad (1.6)$$

In equation (1.6), the function κ is a analytic multi-valued function, defined in (2.4); the functions ψ_{\pm} are some particular solutions of equation

$$H_0\psi = (E - W(\varphi))\psi,$$

analytic in φ on these canonical domains and called Bloch solutions. We will prove the existence of such functions in section 5.2.

A. Fedotov and F. Klopp prove the existence of functions with standard asymptotic only on compact domains of the complex plane. We shall extend some results on infinite strips of the complex plane. The consistency condition implies that the function h_-^g satisfies the standard asymptotic (1.6) to the left of $-A$ and that h_+^d satisfies an analogous property to the right of A . Thus, the computation of $w(h_-^g, h_+^d)$ is similar to the calculations of A. Fedotov and F. Klopp. We must find a sufficiently large domain of the complex plane, in which we know the Wronskian of h_-^g and h_+^d .

The methods used in their works underline some topological obstacles, which change the standard asymptotic (1.6); these obstacles depend on W and E . We give precise assumptions in sections 2.2 and 2.4.

2 The main results

In this section, we describe the general context and the main results of the paper.

First, we present the assumptions on the potentials V and W , and on the interval J . There are mainly three kinds of assumptions. Firstly, the study requires some assumptions on the decay of W to develop the scattering theory. Then, in view of the hypotheses of the complex WKB method of [6], we assume that W is analytic in some domain of the complex plane. Finally, we shall depict the geometric framework and particularly the subset $(E - W)^{-1}(\mathbb{R})$.

We obtain an equation for the eigenvalues in terms of geometric objects depending on H_0 , W and E : the phases and action integrals, defined in sections 2.5.

2.1 The potential V

We assume that V has the following properties:

(H_{V,p}) V is L^2_{loc} , 1-periodic.

We consider (1.3) as a perturbation of the periodic equation:

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = (E - W(\varphi))\psi(x). \quad (2.1)$$

We shall use some well known facts about periodic Schrödinger operators. They are described in detail in section 4.

We just recall elementary results on H_0 . The operator H_0 defined in (1.2) is a self-adjoint operator on $H^2(\mathbb{R})$. The spectrum of H_0 consists of intervals of the real axis:

$$\sigma(H_0) = \bigcup_{n \in \mathbb{N}} [E_{2n+1}, E_{2n+2}], \quad (2.2)$$

such that:

$$E_1 < E_2 \leq E_3 < E_4 \dots E_{2n} \leq E_{2n+1} < E_{2n+2} \dots, \quad E_n \rightarrow +\infty, n \rightarrow +\infty.$$

These intervals $[E_{2n+1}, E_{2n+2}]$ are called the *spectral bands*. We set $E_0 = -\infty$. The intervals (E_{2n}, E_{2n+1}) are called the *spectral gaps*. If $E_{2n} \neq E_{2n+1}$, we say that the gap is open.

Furthermore, we assume that V satisfies:

(H_{V,g}) Every gap of H_0 is not empty.

This assumption is “generic”, we refer to [19] section XIII.16. An important object of the theory of one-dimensional periodic operators is the Bloch quasi-momentum k (see section 4). This function is a multi-valued analytic function; its branch points are the ends of the spectrum, they are of square root type. We shall give a few details about this function in section 4. Finally, we suppose:

(H_V) V satisfy $(H_{V,p})$ and $(H_{V,g})$.

2.2 The perturbation W

2.2.1 Smoothness assumptions

We assume that W is such that:

$(H_{W,r})$ There exists $Y > 0$ such that W is analytic in the strip $S_Y = \{|\text{Im } (\xi)| \leq Y\}$ and there exists $s > 1$ et $C > 0$ such that for $z \in S_Y$, we have:

$$|W(z)| \leq \frac{C}{1 + |z|^s}. \quad (2.3)$$

These assumptions are essential to develop the complex WKB method. The analyticity of the perturbation is crucial in the theory of [6]. The decay of W replaces the compactness resulting from periodicity in [6]. We begin with presenting the complex momentum. This main object of the complex WKB method shows the importance of $W^{-1}(\mathbb{R})$.

2.2.2 The complex momentum and its branch points

We put:

$$\mathbb{C}_+ = \{\varphi \in \mathbb{C} ; \text{Im } \varphi \geq 0\} \text{ and } \mathbb{C}_- = \{\varphi \in \mathbb{C} ; \text{Im } \varphi \leq 0\}.$$

For equation (1.3), we consider the analytic function κ defined by

$$\kappa(\varphi) = k(E - W(\varphi)). \quad (2.4)$$

We recall that the function k is presented in section 2.1. The function κ is called the complex momentum. It plays a crucial role in adiabatically perturbed problems, see [2, 6].

\mathbb{N} is the set of non-negative integers. We define:

$$\Upsilon(E) = \{\varphi \in S_Y ; \exists n \in \mathbb{N}^* / E - W(\varphi) = E_n\} \quad (2.5)$$

The set of branch points of κ is clearly a subset of $\Upsilon(E)$. The following result gives a characterization of the branch points of κ among the points of $\Upsilon(E)$:

Lemma 2.1. *Let φ be a point of $\Upsilon(E)$. If $\inf\{q ; W^{(q)}(\varphi) \neq 0\} \in 2\mathbb{N} + 1$, then φ is a branch point of κ .*

This result follows from the fact that the ends of the spectrum are of square root type.

2.2.3 Geometric assumptions

The spectrum $\sigma(H_0)$ consists of real intervals. Fix $E \in \mathbb{R}$. If $E - W(\varphi)$ is in the spectrum $\sigma(H_0)$, then $W(\varphi)$ is real. The spectral study of (1.3) is then tightly connected with the geometry of $W^{-1}(\mathbb{R})$.

We state now the geometric assumptions for W . These assumptions are mainly a description of $W^{-1}(\mathbb{R})$ in a strip containing the real axis. We call strictly vertical a line whose slope does not vanish; for precise definitions, we refer to section 5.1.1.

$(H_{W,g})$

1. $W|_{\mathbb{R}}$ is real and has a finite number of extrema, which are non-degenerate.

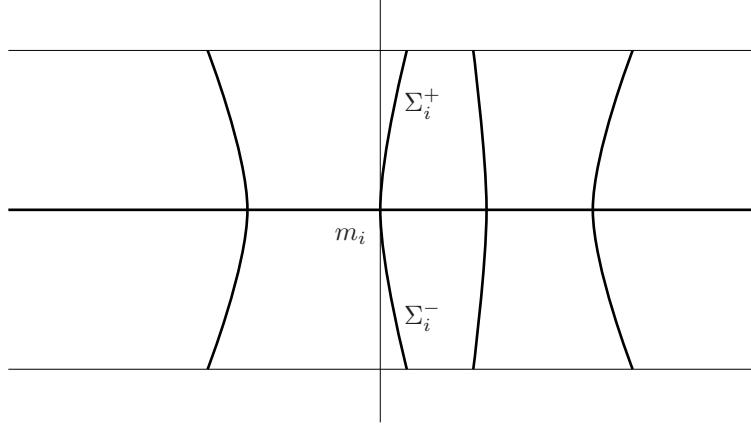


Figure 1: A subset of $W^{-1}(\mathbb{R})$

2. There exists $Y > 0$ and a finite sequence of strictly vertical lines containing an extremum of W , such that:

$$W^{-1}(\mathbb{R}) \cap S_Y = \bigcup_{i \in \{1 \dots p\}} (\Sigma_i) \cup \mathbb{R}. \quad (2.6)$$

2.3 Some remarks

- Since W is real analytic, we know that $W(\bar{\varphi}) = \overline{W(\varphi)}$; this implies that $W^{-1}(\mathbb{R})$ is symmetric with respect to the real axis.
- We define $\Sigma_i^+ = \Sigma_i \cap \mathbb{C}_+$ and $\Sigma_i^- = \Sigma_i \cap \mathbb{C}_-$.
- Figure 1 shows an example of the pre-image of the real axis by such a potential.

As we have explained in section 1.3, we cover the strip S_Y with local canonical domains. On these domains, we construct consistent functions with standard behavior (ie satisfying (1.5) and (1.6)).

To compute the connection between the bases associated with different domains, we get round the branch points (for analog studies, we refer the reader to [11, 8]). We will now state some more accurate assumptions about the configuration of the branch points; in particular, these assumptions specify $(E - W)^{-1}(\sigma(H_0))$ when E is real. The spectral results of A. Fedotov and F. Klopp on perturbed periodic equation have shown the importance of the relative positions of J and $\sigma(H_0)$.

2.4 Assumptions on the interval J

Now, we describe the interval J on which we study equation (1.3).

2.4.1 Hypotheses

We assume that the interval J is a compact interval satisfying:

(H_J)

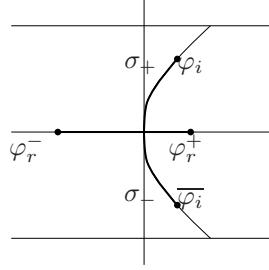


Figure 2: $(E - W)^{-1}(\sigma(H_0))$

1. For any $E \in J$, there exists only one band B of $\sigma(H_0)$ such that the pre-image $C := (E - W)^{-1}(B)$ is not empty.
2. For any $E \in J$, $C := (E - W)^{-1}(B)$ is connected and compact and $(E - W)^{-1}(B)$ contains exactly one real extremum of W .

2.4.2 Consequences

- (H_J) implies that J is included in a gap.
- The band B in (H_J) (1) depends a priori on E . But, since J is connected, the band B is fixed for any $E \in J$.
- Similarly, the extremum of W in assumption (H_J) (2) depends on E , but by connectedness, it is the same for any $E \in J$.

2.4.3 Notations

Put $B = [E_{2n-1}, E_{2n}]$, for $n \in \mathbb{N}^*$. Moreover, we can always change W or φ so that the extremum of W in (2) is 0.

Then (H_J) has the following consequences:

1. For any $E \in J$, $(E - W)^{-1}(\sigma(H_0)) \cap S_Y = (E - W)^{-1}(B) \cap S_Y$
2. Let $E_r \in \{E_{2n-1}, E_{2n}\}$ be the end of B satisfying $E_r \in (E - W)(\mathbb{R})$ for any $E \in J$. We define E_i such that $\{E_i, E_r\} = \{E_{2n-1}, E_{2n}\}$.
3. There are exactly four branch points $(\varphi_r^-, \varphi_r^+) \in \mathbb{R}^2$ and (φ_i, φ_i^-) in S_Y related to E_r and E_i . They satisfy:

$$E - W(\varphi_r^+) = E_r, E - W(\varphi_r^-) = E_r, \varphi_r^- < 0 < \varphi_r^+,$$

$$E - W(\varphi_i) = E - W(\varphi_i^-) = E_i, \operatorname{Im} \varphi_i > 0.$$

4. There exists a strictly vertical line σ containing 0 and connecting φ_i^- to φ_i , such that $(E - W)^{-1}(B) \cap S_Y = [\varphi_r^-, \varphi_r^+] \cup \sigma$. We define $\sigma_+ = \sigma \cap \mathbb{C}_+$ and $\sigma_- = \sigma \cap \mathbb{C}_-$. We let $\Sigma = (E - W)^{-1}(\mathbb{R}) \setminus \mathbb{R}$, $\sigma \subset \Sigma$.

These objects are described in figure 2.

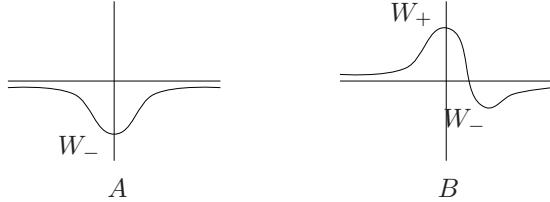


Figure 3: Some examples of potential W

2.4.4 Remarks and examples

We first give a few comments on assumption (H_J) .

- We call C the cross.
- This assumption means intuitively that, in S_Y , we see the band B only near the extremum 0.

To illustrate these technical assumptions, we give a few examples of potentials W and intervals J . We have depicted some examples in figure 3.

- The simplest case is when W has only a non-degenerate minimum W_- (see figure 3 A). in concrete terms, we can think of the example:

$$W(x) = \frac{-\alpha}{1+x^2}, \quad \alpha > 0,$$

Then, if we fix $B = [E_{2n-1}, E_{2n}]$ and $Y < 1$, we can choose $J = [a, b]$ such that:

$$\max\{E_{2n-2}, E_{2n-1} - \alpha, E_{2n} - \frac{\alpha}{1-Y^2}\} < a < b < \min\{E_{2n-1}, E_{2n} - \alpha, E_{2n+1} - \frac{\alpha}{1-Y^2}\}$$

- We can assume that W has a maximum W_+ and a minimum W_- , if J is chosen to see the band only near the maximum (see figure 3 B).

$$W(x) = \frac{2}{1+x^2} - \frac{1}{1+(x-5)^2}$$

$$J \subset]E_{2n-1} + W_+, E_{2n-2} + W_+ [\cup]E_{2n}, E_{2n+1} + W_- [, \quad |J| \leq |E_{2n-2} - E_{2n-1}|$$

Consider this example a little further. The choice of Y is more complicated in this case. The study of equation $W(u) = w$ for $w > W_+$ shows that there exists only one solution in the strip $\{\text{Im } u \in]0, 1[\}$ that we call $Z(w)$; we choose $Y \in \sup_{E \in J} Z(E - E_{2l-1}), \inf_{E \in J} Z(E - E_{2l-2})[$.

- In fact, we could adapt our method to weaker assumptions. For example, we can assume that we do not see the branch points φ_i and $\overline{\varphi_i}$ (incomplete cross), which means that the vertical line σ does not contain any branch points of κ . We refer to section 2.5.4 for some details.
- For the sake of simplicity, we have assumed that all the extrema of W are non degenerate. Actually, it suffices to assume that only the extremum of W in 0 is non degenerate.
- Similarly, we could weaken assumption $(H_{V,g})$. We only have to assume that the gaps adjoining the band B of (H_J) are not empty.

2.5 Phases and action

In this section, we define the tunneling coefficient t and the phases Φ et Φ_d ; these analytic objects play an essential role in the location of the eigenvalues. These coefficients are represented as integrals of the complex momentum κ in the φ plane.

In the strip S_Y , we consider κ a branch of the complex momentum, continuous on C .

2.5.1 Definition and properties

We introduce the action S and the phases Φ and Φ_d related to the branch κ .

Definition 2.1. *We define the phase:*

$$\Phi(E) = \int_{\varphi_r^-}^{\varphi_r^+} (\kappa(u) - \kappa(\varphi_r^-)) du, \quad (2.7)$$

the action:

$$S(E) = i \int_{\sigma} (\kappa(u) - \kappa(\varphi_i)) du, \quad (2.8)$$

the second phase:

$$\Phi_d(E) = \int_{\varphi_r^-}^0 (\kappa(u) - \kappa(\varphi_r^-)) du + \int_{\varphi_r^+}^0 (\kappa(u) - \kappa(\varphi_r^+)) du + \int_{\sigma_+} (\kappa(u) - \kappa(\varphi_i)) du - \int_{\sigma_-} (\kappa(u) - \kappa(\overline{\varphi_i})) du. \quad (2.9)$$

In section 8, we prove the following result on the behavior of the coefficients Φ , S and Φ_d .

Lemma 2.2. *There exists a branch $\tilde{\kappa}_i$ such that the phases and action integrals have the following properties:*

1. Φ , S , Φ_d are analytic in E in a complex neighborhood of the interval J .
2. Φ , S , Φ_d take real values on J . Φ and S are positive on J .
3. $\forall E \in J$, $\Phi'(E)(E_i - E_r) > 0$, $S(E) \leq 2\pi \operatorname{Im}(\varphi_i(E))$.

We define *the tunneling coefficient* :

$$t(E, \varepsilon) = \exp(-S(E)/\varepsilon). \quad (2.10)$$

t is exponentially small.

2.5.2 Remark

The phase and action are simply a generalization of the coefficients of the form $\int \sqrt{E - W(\varphi)} d\varphi$, well-known in the case $V = 0$ (we refer to [5, 11, 18]).

We point out that the coefficient Φ depend only on the value of W on the real axis, whereas S and Φ_d depend on the values of W in the complex plane. The phase Φ is independent of the analyticity of W unlike S and Φ_d .

Now, we state the equation for eigenvalues for (1.3).

2.5.3 The main result

Theorem 1. *Equation for eigenvalues.*

Let V , W and J satisfy assumptions (H_V) , $(H_{W,r})$, $(H_{W,g})$ and (H_J) . Fix $Y_0 \in]0, Y[$.

There exists a complex neighborhood \mathcal{V} of J , a real number $\varepsilon_0 > 0$ and two functions $\tilde{\Phi}$ and $\tilde{\Phi}_d$ with complex values, defined on $\mathcal{V} \times]0, \varepsilon_0[$ such that:

- The functions $\tilde{\Phi}(\cdot, \varepsilon)$ and $\tilde{\Phi}_d(\cdot, \varepsilon)$ are analytic on \mathcal{V} . Moreover, $\tilde{\Phi}$ and $\tilde{\Phi}_d$ satisfy:

$$\tilde{\Phi}(E, \varepsilon) = \Phi(E) + h_0(E, \varepsilon) \quad \text{and} \quad \tilde{\Phi}_d(E, \varepsilon) = \Phi_d(E) + h_1(E, \varepsilon),$$

where ρ is a real coefficient, $h_0(E, \varepsilon) = o(\varepsilon)$ and $h_1(E, \varepsilon) = o(\varepsilon)$ uniformly in $E \in \mathcal{V}$.

- If we define the energy levels $\{E^{(l)}(\varepsilon)\}$ in J by:

$$\frac{\tilde{\Phi}(E^{(l)}(\varepsilon), \varepsilon)}{\varepsilon} = l\pi + \frac{\pi}{2}, \quad \forall l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon)\}, \quad (2.11)$$

then, for any $\varepsilon \in]0, \varepsilon_0[$,

- the spectrum of $H_{\varphi, \varepsilon}$ in J consists in a finite number of eigenvalues, that is to say

$$\sigma(H_{\varphi, \varepsilon}) \cap J = \bigcup_{l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon)\}} \{E_l(\varphi, \varepsilon)\}, \quad (2.12)$$

- these eigenvalues satisfy

$$E_l(\varphi, \varepsilon) = E^{(l)}(\varepsilon) + \varepsilon(-1)^{l+1} \frac{t(E^{(l)}(\varepsilon), \varepsilon)}{\Phi'(E^{(l)}(\varepsilon))} \left[\cos \left(\frac{\tilde{\Phi}_d(E^{(l)}(\varepsilon), \varepsilon) + 2\pi\varphi + \rho\varepsilon}{\varepsilon} \right) + t(E^{(l)}(\varepsilon), \varepsilon)r(E^{(l)}(\varepsilon), \varphi, \varepsilon) \right], \quad (2.13)$$

where there exists $c > 0$ such that

$$\sup_{E \in \mathcal{V}, \varphi \in \mathbb{R}} r(E, \varphi, \varepsilon) < \frac{1}{c} e^{-\frac{c}{\varepsilon}}.$$

We prove this result in section 8.

2.5.4 Remark

If we only assume that σ does not contain any branch points, asymptotic (2.13) is replaced by the estimate:

$$|E_l(\varphi, \varepsilon) - E^{(l)}(\varepsilon)| < C e^{-\frac{2\pi Y}{\varepsilon}}$$

where $2Y$ is the width of the strip S_Y .

2.5.5 Application : asymptotic expansion of the trace

By using the previous result, we can compute the first terms in the asymptotic expansion of the trace formula, and partially recover a result of [4].

Corollary 1. Let $f \in C_0^\infty(\mathbb{R})$ be a real function such that $\text{Supp } f \in J$. Then the function $f(H_{\varphi, \varepsilon})$ is ε -periodic in φ and its Fourier expansion satisfies:

$$\text{tr}[f(H_{\varphi, \varepsilon})] = \frac{1}{\varepsilon} \int_0^\varepsilon [f(H_{u, \varepsilon})] du + O(e^{-S/\varepsilon}) \quad (2.14)$$

$$\int_0^\varepsilon [f(H_{u, \varepsilon})] du = \frac{1}{2\pi} \int_{\mathbb{R}_u} \int_{[-\pi, \pi]} f(W(u) + E(\kappa)) d\kappa du + o(\varepsilon) \quad (2.15)$$

where $S = \inf_{e \in \text{Supp } f} S(e) > 0$

We give more details and the proof of this corollary in section 8.6.

3 Main steps of the study

Here, we explain the main ideas of the paper.

3.1 One-dimensional perturbed periodic operators

3.1.1

We consider equation (1.3) as a perturbation of the periodic equation

$$H_0\psi = E\psi \quad (3.1)$$

where the operator H_0 is defined in (1.2). To do that, we shall describe the spectral theory of periodic operators in section 4.

- For the moment, we simply introduce the Bloch solutions of equation (2.1). We call a *Bloch solution* of (2.1) a function Ψ satisfying (2.1) and:

$$\forall x \in \mathbb{R}, \quad \Psi(x+1, E) = \lambda(E)\Psi(x, E), \quad (3.2)$$

with $\lambda \neq 0$ independent of x . The coefficient $\lambda(E)$ is called *Floquet multiplier*. We represent $\lambda(E)$ in the form $\lambda(E) = e^{ik(E)}$; k is the quasi-momentum presented in section 2.1 and described in section 4. If $E \notin \sigma(H_0)$, there exist two linearly independent Bloch solutions of (3.1) (see section 4.1). We call them $\tilde{\Psi}_+$ et $\tilde{\Psi}_-$; the associated Floquet multipliers are inverse of each other and the functions $\tilde{\Psi}_\pm$ are represented in the form:

$$\tilde{\Psi}_\pm(x, E) = e^{\pm ik(E)x} p_\pm(x, E) \quad \text{avec} \quad p_\pm(x+1, E) = p_\pm(x, E).$$

For $\text{Im } k(E) > 0$, the function $\tilde{\Psi}_+(x, E)$ tends to 0 as x tends to $+\infty$ and the function $\tilde{\Psi}_-(x, E)$ tends to 0 as x tends to $-\infty$. Actually, equation (3.2) defines the functions $\tilde{\Psi}_+$ and $\tilde{\Psi}_-$ except for a multiplicative coefficient. Precisely, equation (3.2) defines two one-dimensional vector spaces that we call *Bloch sub-spaces*.

To study the eigenvalues of perturbations of periodic operators, [10] and [17] introduce, for $\text{Im } k(E) > 0$, two functions $(x, \varphi, E, \varepsilon) \mapsto F_+(x, \varphi, E, \varepsilon)$ and $(x, \varphi, E, \varepsilon) \mapsto F_-(x, \varphi, E, \varepsilon)$ solutions of (1.3) satisfying:

$$\lim_{x \rightarrow +\infty} [F_+(x, \varphi, E, \varepsilon) - \tilde{\Psi}_+(x, E)] = 0, \quad \lim_{x \rightarrow -\infty} [F_-(x, \varphi, E, \varepsilon) - \tilde{\Psi}_-(x, E)] = 0 \quad (3.3)$$

Condition (3.3) guarantees the uniqueness of F_+ (resp. of F_-) since the function $\tilde{\Psi}_+$ (resp. $\tilde{\Psi}_-$) tends to 0 as x tends to $+\infty$ (resp. $-\infty$). These functions are called Jost functions; they are generally constructed as solutions of a Lippman-Schwinger integral equation. This construction is an adaptation of the usual theory of scattering (chapter XI of [20]) for a perturbation of laplacian; it consists in looking for particular solutions of (1.3) from the solutions of the periodic equation.

We call *Jost sub-spaces* the sub-spaces \mathcal{J}_+ and \mathcal{J}_- generated by F_+ and F_- .

\mathcal{J}_+ (resp \mathcal{J}_-) is the set of solutions of (1.3) being a member of $L^2([0, \infty))$ (resp. $L^2((-\infty, 0])$).

- Let f and g be two derivable functions, the *Wronskian* of f and g called $w(f, g)$ is defined by:

$$w(f, g) = f'g - fg' \quad (3.4)$$

We recall that if f and g are the solutions of a second-order differential equation, their Wronskian is independent of x . The spectral interest of the Jost sub-spaces is the following:

Proposition 1. *We assume that $\text{Im } k(E) > 0$. Let $h_g^- \in \mathcal{J}_-$ and $h_d^+ \in \mathcal{J}_+$ be two nontrivial Jost solutions of (1.3). E is an eigenvalue of $H_{\varphi, \varepsilon}$ if and only if:*

$$w(h_d^+, h_g^-) = 0 \quad (3.5)$$

To compute the eigenvalues, it suffices to construct the Jost sub-spaces.

3.2 Construction of consistent Jost solutions

We denote by (H_J^0) the following assumption:

(H_J⁰) There exists $n \in \mathbb{N}$ such that \mathbf{J} is a compact interval of $[\mathbf{E}_{2n}, \mathbf{E}_{2n+1}]$.

Clearly, (H_J^0) is weaker than (H_J) .

We introduce a new notation.

For a function $f : \mathcal{U} \subset \mathbb{C}^n \rightarrow \mathbb{C}^p$, we define the function $f^* : \overline{\mathcal{U}} \rightarrow \mathbb{C}^p$:

$$f^*(Z) = \overline{f(\overline{Z})}. \quad (3.6)$$

As we have explained in section 1, an useful idea to study (1.3) is the construction of consistent solutions, i.e. satisfying (1.5). First, we choose in \mathcal{J}_- and \mathcal{J}_+ some consistent bases. We shall prove the following result:

Theorem 2. *We assume that (H_V) , $(H_{W,r})$ and (H_J^0) are satisfied. Fix $X > 1$. Then, there exist a complex neighborhood $\mathcal{V} = \overline{\mathcal{V}}$ of J , a real $\varepsilon_0 > 0$, two points m_g and m_d in \mathbb{C} , two real numbers A_g and A_d and two functions $(x, \varphi, E, \varepsilon) \mapsto h_-^g(x, \varphi, E, \varepsilon)$, $(x, \varphi, E, \varepsilon) \mapsto h_+^d(x, \varphi, E, \varepsilon)$ such that:*

- The functions $(x, \varphi, E, \varepsilon) \mapsto h_-^g(x, \varphi, E, \varepsilon)$ and $(x, \varphi, E, \varepsilon) \mapsto h_+^d(x, \varphi, E, \varepsilon)$ are defined and consistent on $\mathbb{R} \times S_Y \times \mathcal{V} \times]0, \varepsilon_0[$.
- For any $x \in [-X, X]$ and $\varepsilon \in]0, \varepsilon_0[$, $(\varphi, E) \mapsto h_-^g(x, \varphi, E, \varepsilon)$ and $(\varphi, E) \mapsto h_+^d(x, \varphi, E, \varepsilon)$ are analytic on $S_Y \times \mathcal{V}$.
- The function $x \mapsto h_-^g(x, \varphi, E, \varepsilon)$ (resp. $x \mapsto h_+^d(x, \varphi, E, \varepsilon)$) is a basis of \mathcal{J}_- (resp. \mathcal{J}_+).
- The functions h_-^g and h_+^d have the following asymptotic behavior:

$$h_-^g(x, \varphi, E, \varepsilon) = e^{\frac{-i}{\varepsilon} \int_{m_g}^{\varphi} \kappa(u) du} \psi_-(x, \varphi, E)(1 + R_g(x, \varphi, E, \varepsilon)), \quad (3.7)$$

and

$$h_+^d(x, \varphi, E, \varepsilon) = e^{\frac{i}{\varepsilon} \int_{m_d}^{\varphi} \kappa(u) du} \psi_+(x, \varphi, E)(1 + R_d(x, \varphi, E, \varepsilon)), \quad (3.8)$$

where

– R_g and R_d satisfy:

$$\sup_{x \in [-X, X], Re \varphi < A_g, E \in \mathcal{V}} \max\{|R_g(x, \varphi, E, \varepsilon)|, |\partial_x R_g(x, \varphi, E, \varepsilon)|\} \leq r(\varepsilon).$$

$$\sup_{x \in [-X, X], Re \varphi > A_d, E \in \mathcal{V}} \max\{|R_d(x, \varphi, E, \varepsilon)|, |\partial_x R_d(x, \varphi, E, \varepsilon)|\} \leq r(\varepsilon),$$

with

$$\lim_{\varepsilon \rightarrow 0} r(\varepsilon) = 0.$$

- The functions ψ_+ and ψ_- are the Bloch canonical solutions of the periodic equation (3.1) defined in section 5.2.
- There exist two real numbers $\sigma_g \in \{-1, 1\}$, $\sigma_d \in \{-1, 1\}$, an integer p and two functions $E \mapsto \alpha_g(E)$ and $E \mapsto \alpha_d(E)$ such that:
 1. For any $\varepsilon \in]0, \varepsilon_0[$, $x \in \mathbb{R}$, $E \in \mathcal{V}$, et $\varphi \in S_Y$, we have:

$$\alpha_g^*(E)(h_-^g)^*(x, \varphi, E, \varepsilon) = i\sigma_g e^{-\frac{i}{\varepsilon} 2p\pi x} \alpha_g(E) h_-^g(x, \varphi, E, \varepsilon) \quad (3.9)$$

$$\alpha_d^*(E)(h_+^d)^*(x, \varphi, E, \varepsilon) = i\sigma_d e^{\frac{i}{\varepsilon} 2p\pi x} \alpha_d(E) h_+^d(x, \varphi, E, \varepsilon) \quad (3.10)$$

2. The functions α_g and α_d are analytic and given by (6.22) and (6.25). They do not vanish on \mathcal{V} .

We immediately deduce from Theorem 2 and Proposition 1 that the eigenvalues of $H_{\varphi,\varepsilon}$ are characterized by:

$$w(h_-^g(\cdot, \varphi, E, \varepsilon), h_+^d(\cdot, \varphi, E, \varepsilon)) = 0 \quad (3.11)$$

Theorem 2 is the consequence of two main ideas:

- First, we adapt the construction of Jost functions developed by [10, 16]. Indeed, this construction proves that asymptotic (6.1) is only satisfied on domains which depend on ε (see section 6).
- We must understand how this asymptotic evolves on a domain which does not depend on ε . To do that, we extend the continuation results of Fedotov and Klopp in a non compact frame (see section 7).

3.3 Conclusion

To finish the computations, it suffices to apply the methods of [6]. We have to go through the cross (see figure 2). We will show that there exists in the neighborhood of the cross a consistent basis f_\pm^i with standard asymptotic. We shall express the functions h_-^g and h_+^d on this basis (section 8).

4 Periodic Schrödinger operators on the real line

We now discuss the periodic operator (1.2) where V is a 1-periodic, real-valued, $\mathbf{L}^2_{\text{loc}}$ -function. We collect known results needed in the present paper (see [15, 14, 25]).

4.1 Bloch solutions

Let $\tilde{\Psi}$ be a solution of the equation

$$H_0 \tilde{\Psi} = \mathcal{E} \tilde{\Psi} \quad (4.1)$$

satisfying the relation

$$\tilde{\Psi}(x+1) = \lambda \tilde{\Psi}(x), \quad \forall x \in \mathbb{R} \quad (4.2)$$

for some complex number $\lambda \neq 0$ independent of x . Such a solution is called a *Bloch solution*, and the number λ is called the *Floquet multiplier*. Let us discuss the analytic properties of Bloch solutions.

In (2.2), we have denoted by $[E_1, E_2], \dots, [E_{2n+1}, E_{2n+2}], \dots$ the spectral bands of the periodic Schrödinger equation. Consider Γ_\pm two copies of the complex plane $\mathcal{E} \in \mathbb{C}$ cut along the spectral bands. Paste them together to get a Riemann surface with square root branch points. We denote this Riemann surface by Γ .

One can construct a Bloch solution $\tilde{\Psi}(x, \mathcal{E})$ of equation (4.1) meromorphic on Γ . The poles of this solution are located in the spectral gaps. Precisely, each spectral gap contains precisely one simple pole. This pole is situated either on Γ_+ or on Γ_- . The position of the pole is independent of x . For the details, we refer to [10].

Except at the edges of the spectrum (i.e. the branch points of Γ), the restrictions $\tilde{\Psi}_\pm$ of $\tilde{\Psi}$ on Γ_\pm are linearly independent solutions of 4.1. Along the gaps, these functions are real and satisfy :

$$\overline{\tilde{\Psi}_\pm(x, \mathcal{E} - i0)} = \tilde{\Psi}_\pm(x, \mathcal{E} + i0), \quad \forall \mathcal{E} \in]E_{2n}, E_{2n+1}[, \quad n \in \mathbb{N}. \quad (4.3)$$

Along the bands, we have :

$$\overline{\tilde{\Psi}_\pm(x, \mathcal{E} - i0)} = \tilde{\Psi}_\mp(x, \mathcal{E} + i0), \quad \forall \mathcal{E} \in]E_{2n+1}, E_{2n+2}[, \quad n \in \mathbb{N}. \quad (4.4)$$

4.2 Bloch quasi-momentum

4.2.1

Consider the Bloch solution $\tilde{\Psi}(x, \mathcal{E})$ introduced in the previous subsection. The corresponding Floquet multiplier $\lambda(\mathcal{E})$ is analytic on Γ . Represent it in the form:

$$\lambda(\mathcal{E}) = \exp(ik(\mathcal{E})). \quad (4.5)$$

The function $k(\mathcal{E})$ is called Bloch quasi-momentum. It has the same branch points as $\tilde{\Psi}(x, \mathcal{E})$, but the corresponding Riemann surface is more complicated.

To describe the main properties of k , consider the complex plane cut along the real line from E_1 to $+\infty$. Denote the cut plane by \mathbb{C}_0 . One can fix there a single valued branch of the quasi-momentum by the condition

$$ik_0(\mathcal{E}) < 0, \quad \mathcal{E} < E_1. \quad (4.6)$$

All the other branches of the quasi-momentum have the form $\pm k_0(\mathcal{E}) + 2\pi m, m \in \mathbb{Z}$. The \pm and the number m are indexing these branches. The image of \mathbb{C}_0 by k_0 is located in the upper half of the complex plane.

$$\operatorname{Im} k_0(\mathcal{E}) > 0, \quad \mathcal{E} \in \mathbb{C}_0. \quad (4.7)$$

In figure 4, we drew several curves in \mathbb{C}_0 and their images under transformation $E \mapsto k_0(E)$. The quasi-momentum $k_0(E)$ is real along the spectral zones, and, along the spectral gaps, its real part is constant; in particular, we have

$$k_0(E_1) = 0 \quad k_0(E_{2l} \pm i0) = k_0(E_{2l+1} \pm i0) = \pm\pi l, \quad l \in \mathbb{N}. \quad (4.8)$$

All the branch points of k are of square root type. Let E_m be one of the branch points of k . Then, each function:

$$f_m^\pm(\mathcal{E}) = (k_0(\mathcal{E} \pm i0) - k_0(E_m \pm i0)) / \sqrt{\mathcal{E} - E_m}, \quad E \in \mathbb{R} \quad (4.9)$$

can be analytically continued in a small vicinity of the branch point E_m .

Finally, we note that

$$k_0(\mathcal{E}) = \sqrt{\mathcal{E}} + O(1/\sqrt{\mathcal{E}}), \quad |\mathcal{E}| \rightarrow \infty \quad (4.10)$$

where $E \in \mathbb{C}_0$ and $0 < \arg E < 2\pi$.

The values of the quasi-momentum k_0 on the two sides of the cut $[E_1, +\infty)$ are related to each other by the formula:

$$\forall \mathcal{E} \in]E_1, +\infty[, \quad k_0(\mathcal{E} + i0) = -\overline{k_0(\mathcal{E} - i0)}, \quad E_1 \leq \mathcal{E}. \quad (4.11)$$

Consider the spectral gap $(E_{2l}, E_{2l+1}), l \in \mathbb{N}$. Let \mathbb{C}_l be the complex plane cut from $-\infty$ to E_{2l} and from E_{2l+1} to $+\infty$. Denote by k_l the branch of the quasi-momentum analytic on \mathbb{C}_l and coinciding with k_0 for $\operatorname{Im} E > 0$. Then, one has:

$$\forall \mathcal{E} \in]-\infty, E_{2l}[\cup]E_{2l+1}, +\infty[, \quad k_l(\mathcal{E} + i0) + \overline{k_l(\mathcal{E} - i0)} = 2\pi l. \quad (4.12)$$

4.3 Periodic components of the Bloch solution

Let D a simply connected domain that does not contain any branch point of k . On D , we fix an analytic branch of k . Consider two copies of D , denoted by D_\pm , corresponding to two sheets of \mathcal{G} . Now we redefine $\tilde{\Psi}_\pm$ to be the restrictions of $\tilde{\Psi}$ to D_\pm . They can be represented in the form:

$$\tilde{\Psi}_\pm(x, \mathcal{E}) = e^{\pm ik(\mathcal{E})x} p_\pm(x, \mathcal{E}), \quad \mathcal{E} \in D \quad (4.13)$$

where $p_l^\pm(x, \mathcal{E})$ are 1-periodic in x ,

$$p_\pm(x+1, \mathcal{E}) = p_\pm(x, \mathcal{E}), \quad \forall x \in \mathbb{R} \quad (4.14)$$

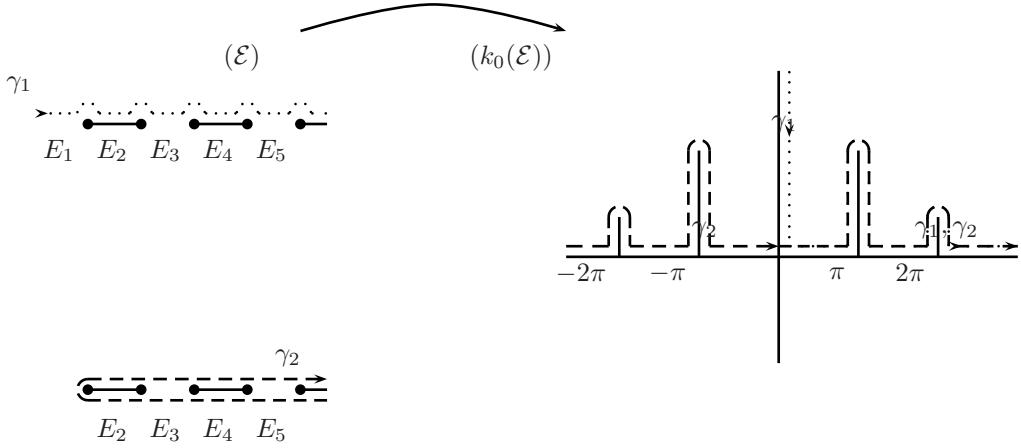


Figure 4: The quasi-momentum k

4.4 Analytic solutions of (4.1)

To describe the asymptotic formulas of the complex WKB method for equation (1.3), one needs specially normalized Bloch solutions of the equation (4.1).

Let \mathcal{D} be a simply connected domain in the complex plane containing no branch point of the quasi-momentum k . We fix on \mathcal{D} a continuous determination of k . We fix $\mathcal{E}_0 \in \mathcal{D}$. We recall the following result ([6, 9]).

Lemma 4.1. *We define the functions g_{\pm} :*

$$g_{\pm} : \mathcal{D} \rightarrow \mathbb{C} ; \mathcal{E} \mapsto -\frac{\int_0^1 p_{\mp}(x, \mathcal{E}) \partial_{\mathcal{E}} p_{\pm}(x, \mathcal{E}) dx}{\int_0^1 p_+(x, \mathcal{E}) p_-(x, \mathcal{E}) dx}, \quad (4.15)$$

and the functions ψ_{\pm}^0 :

$$\psi_{\pm}^0 : \mathbb{R} \times \mathcal{D} \rightarrow \mathbb{C} ; (x, \mathcal{E}) \mapsto \sqrt{k'(\mathcal{E})} e^{\int_{\mathcal{E}_0}^{\mathcal{E}} g_{\pm}(e) de} \tilde{\Psi}_{\pm}(x, \mathcal{E}). \quad (4.16)$$

The functions $\mathcal{E} \mapsto \psi_{\pm}^0(x, \mathcal{E})$ are analytic on \mathcal{D} , for any $x \in \mathbb{R}$. The functions ψ_{\pm}^0 are called analytic Bloch solutions normalized at the point \mathcal{E}_0 of (4.1).

Sometimes, we shall denote $\psi_{\pm}^0(x, \mathcal{E}, \mathcal{E}_0)$ to specify the normalization. We refer to section 1.4.4 of [9] for the details of the proof. The proof follows from the study of the poles of $\tilde{\Psi}_{\pm}$ and the zeros of k' . The poles of g_{\pm} are simple and exactly situated at the singularities of $\sqrt{k'} \tilde{\Psi}_{\pm}$. The computation of the residues of g_{\pm} at these points completes the proof.

4.5 Useful formulas

We end this section with some useful formulas. We recall that the functions g_{\pm} are given in (4.15). Fix $n \in \mathbb{N}$. Equations (4.3) and (4.4) lead to the following relations:

$$g_{\pm}^*(\mathcal{E}) = g_{\pm}(x, \mathcal{E}), \quad \forall \mathcal{E} \in]E_{2n}, E_{2n+1}[. \quad (4.17)$$

$$g_{\pm}^*(\mathcal{E}) = g_{\mp}(x, \mathcal{E}), \quad \forall \mathcal{E} \in]E_{2n+1}, E_{2n+2}[. \quad (4.18)$$

5 Main tools of the complex WKB method

In this section, we recall the main tools of the complex WKB method on compact domains. The idea of the method is to construct some consistent functions of (1.3) with asymptotic behavior (1.6). This construction is not possible on any domain of the complex plane but on some domains called canonical. We apply the results of [6, 8, 7] to the assumptions $(H_{W,g})$ and (H_J) . We build a neighborhood of the cross, in which we construct a consistent basis with standard behavior (1.6). In this section, we fix Y such that the assumptions $(H_{W,g})$ and (H_J) are satisfied in the strip S_Y .

5.1 Canonical domains

The canonical domain is the main geometric notion of the complex WKB method.

5.1.1 The complex momentum

The canonical domains can be described in terms of the complex momentum $\kappa(\varphi)$. Remind that this function is defined by formula (2.4). We have described κ in section 2.2.2. The properties of κ depend on the spectral parameter E and of the analytic properties of W . We first formulate some definitions ([6]).

5.1.2 Vertical, strictly vertical curves

Definition 5.1. *We say that a curve γ is vertical if it intersects the lines $\text{Im } z = \text{Const}$ at non-zero angles θ .*

We say that a curve γ is strictly vertical if there is a positive number δ such that, at any point of γ , the intersection angle θ satisfies the inequality:

$$\delta < \theta < \pi - \delta. \quad (5.1)$$

5.1.3 Canonical, strictly canonical curves

Let γ be a vertical curve which does not contain any branch point. On γ , fix a continuous branch of the momentum of κ .

Definition 5.2. *We call γ canonical if, along γ ,*

- $\text{Im } \varphi \mapsto \text{Im } \int_{\varphi}^{\varphi'} \kappa(u)du$ is strictly increasing.
- $\text{Im } \varphi \mapsto \text{Im } \int_{\varphi}^{\varphi'} (\kappa(u) - \pi)du$ is strictly decreasing.

Assume that γ is strictly vertical. If there is a positive number δ such that, along γ :

$$\text{Im } \int_{\varphi}^{\varphi'} \kappa(u)du \geq \delta \text{Im } (\varphi' - \varphi) \quad \forall (\varphi, \varphi') \in \gamma^2, \quad (5.2)$$

and

$$\text{Im } \int_{\varphi}^{\varphi'} (\pi - \kappa(u))du \geq \delta \text{Im } (\varphi' - \varphi) \quad \forall (\varphi, \varphi') \in \gamma^2, \quad (5.3)$$

we call γ δ – strictly canonical.

We identify the complex numbers with vectors in \mathbb{R}^2 . To construct canonical lines, we have to study the vector fields κ and $\kappa - \pi$, or rather their integral curves. For $\varphi \in D$, $S(\varphi)$ denotes the sector of apex φ such that, for any vector $z \in S(\varphi)$, we have:

$$\text{Im } (i\overline{\kappa(\varphi)}(z - \varphi)) > 0 \text{ et } \text{Im } (i(\overline{\kappa(\varphi)} - \pi)(z - \varphi)) < 0. \quad (5.4)$$

Let $\gamma \in D$ a curve which does not contain any branch point. For all $\varphi \in \gamma$, we denote $t(\varphi)$ the vector tangent to γ in φ and oriented upward. The curve $\gamma \in D$ is canonical for the determination κ if and only if for any $\varphi \in \gamma$, the vector $t(\varphi)$ belongs to $S(\varphi)$ (see figure 5). The cone $S(\varphi)$ depends on the determination of κ . For example, if κ satisfies $\text{Re } \kappa \in]0, \pi[$, this cone is not empty.

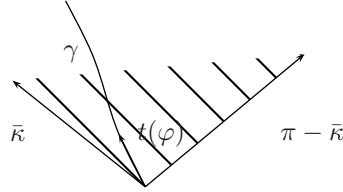


Figure 5: The cone $S(\varphi)$

5.1.4

In what follows, ξ_1 and ξ_2 are two points in \mathbb{C} such that

$$\operatorname{Im} \xi_1 < \operatorname{Im} \xi_2.$$

We shall denote by γ a smooth curve going from ξ_1 to ξ_2 ; this curve will always be oriented from ξ_1 to ξ_2 .

5.1.5 Definition of the canonical domain

Let K be a simply connected domain in $\{\operatorname{Im} \varphi \in [\operatorname{Im} \xi_1, \operatorname{Im} \xi_2]\}$ containing no branch points of the complex momentum. On K , fix a continuous branch κ .

Definition 5.3. *We call K a canonical domain for κ , ξ_1 and ξ_2 if it is the union of curves that are connecting ξ_1 and ξ_2 and that are canonical with respect to κ .*

If there is $\delta > 0$ such that K is a union of δ -strictly canonical curves, we call K δ – strictly canonical.

5.1.6

Assume that K is a canonical domain. Denote by ∂K its boundary. Fix a positive number δ . We call the domain

$$\mathcal{C} = \{z \in K ; \operatorname{dist}(z, \partial K) > \delta\}$$

an admissible sub-domain of K .

Note that the branch points of the complex momentum are outside of \mathcal{C} , at a distance greater than δ .

5.2 Canonical Bloch solutions

To describe the asymptotic formulas of the complex WKB method for equation (1.3), we shall use the analytic Bloch solutions of (4.1), defined in Lemma 4.1 for the parameter $\mathcal{E} = E - W(\varphi)$. Precisely, we consider the unperturbed periodic equation:

$$H_0 \psi = (E - W(\varphi)) \psi. \quad (5.5)$$

5.3

Let D be a simply connected domain in S_Y , containing no branch points of κ . The mapping $\varphi \mapsto E - W(\varphi)$ maps D onto a domain $\mathcal{D} \subset \mathbb{C}$. The domain \mathcal{D} does not contain any branch point of k .

Fix $\varphi_0 \in D$, such that $k'(E - W(\varphi_0)) \neq 0$. In Lemma 4.1, we have built the analytic Bloch solutions $\{\psi_{\pm}^0\}$ of equation (4.1), normalized in $E - W(\varphi_0)$. For $\varphi \in D$, we define:

$$\psi_{\pm}(x, \varphi, E) = \psi_{\pm}^0(x, E - W(\varphi)), \quad \forall u \in \mathbb{R}, \quad \forall \varphi \in D. \quad (5.6)$$

In [6], it is proved that the functions $\varphi \mapsto \psi_{\pm}(x, \varphi, E)$ can be analytically continued to D . ψ_{\pm} are called the *canonical Bloch solutions* of equation (5.5). Sometimes, we shall precise $\psi_{\pm}(x, \varphi, E, \varphi_0)$ to specify the normalization.

We define

$$\omega_{\pm}(\varphi, E) = -W'(\varphi)g_{\pm}(E - W(\varphi)). \quad (5.7)$$

We also define:

$$q(\varphi) = \sqrt{k'(E - W(\varphi))} \quad (5.8)$$

5.4 The consistency relation

5.4.1 Consistent functions and consistent bases

We recall that we say that f is a *consistent* function if it satisfies (1.5).

We say also that a basis $\{f_{\pm}\}$ of solutions of (1.3) is a *consistent basis* if:

- The functions f_+ and f_- are consistent.
- Their Wronskian is independent of φ .

5.4.2 Analyticity and consistency

First, we define the width of a set.

Definition 5.4. Fix $Y_0 > 0$ and $M \subset S_{Y_0}$ a set of points. We define $l(M, Y_0)$:

$$l(M, Y_0) = \inf_{y \in [-Y_0, Y_0]} \sup \{ |Re \varphi - Re \varphi'|; (\varphi, \varphi') \in M^2 \text{ such that } Im \varphi = Im \varphi' = y \} \quad (5.9)$$

$l(M, Y_0)$ is called the width of M in S_{Y_0}

One has:

Lemma 5.1. Fix E . We consider $X > 0$, $\tilde{Y} \in]0, Y[$, $\varepsilon_0 > 0$ and K a complex domain such that $l(K, \tilde{Y}) > \varepsilon_0$. We assume that for any $\varepsilon \in]0, \varepsilon_0[$, $f(\cdot, \varphi, E, \varepsilon)$ is a consistent solution of (1.3) for $\varphi \in K$ and that for any $x \in [-X, X]$, the function $\varphi \mapsto f(x, \varphi, E, \varepsilon)$ is analytic on K . Then, for any $\varepsilon \in]0, \varepsilon_0[$ and any $x \in [-X, X]$, the function $\varphi \mapsto f(x, \varphi, E, \varepsilon)$ is analytic on $S_{\tilde{Y}}$.

This result is proved in [7, 9].

5.5 The theorem of the complex WKB method on a compact domain

In this section, we recall the main result of the complex WKB method.

5.5.1 Standard asymptotic behavior

We briefly introduce the notion of standard asymptotic behavior (see [9]). Speaking about a solution having standard asymptotic behavior, we mean first of all that this solution has the asymptotics (1.6) and other properties that we present now.

Fix $E_0 \in \mathbb{C}$. Let $D \subset \mathbb{C}$ a simply connected domain containing no branch points. Let κ be a branch of the complex momentum continuous in D and ψ_{\pm} the canonical Bloch solutions normalized in $\varphi_0 \in D$. We say that a consistent solution f has standard behavior $f \sim e^{\frac{i}{\varepsilon} \int^{\varphi} \kappa(u) du} \psi_{+}(x, \varphi, E)$, respectively $f \sim e^{-\frac{i}{\varepsilon} \int^{\varphi} \kappa(u) du} \psi_{-}(x, \varphi, E)$ in D if

- there exists a complex neighborhood \mathcal{V}_0 of E_0 and $X > 0$ such that f is a consistent solution of equation (1.3) for any $(x, \varphi, E) \in [-X, X] \times D \times \mathcal{V}_0$;
- for any $x \in [-X, X]$, the function $((\varphi, E) \mapsto f(x, \varphi, E, \varepsilon))$ is analytic on $D \times \mathcal{V}_0$;
- for any A , a sub-admissible domain of D , there is a neighborhood \mathcal{V}_A of E_0 such that

$$\forall (x, \varphi, E) \in [-X, X] \times D \times \mathcal{V}_A, \quad f(x, \varphi, E, \varepsilon) = e^{\frac{i}{\varepsilon} \int^\varphi \kappa(u) du} (\psi_+(x, \varphi, E) + o(1)), \quad \varepsilon \rightarrow 0 \quad (5.10)$$

respectively

$$\forall (x, \varphi, E) \in [-X, X] \times D \times \mathcal{V}_A, \quad f(x, \varphi, E, \varepsilon) = e^{-\frac{i}{\varepsilon} \int^\varphi \kappa(u) du} (\psi_-(x, \varphi, E) + o(1)), \quad \varepsilon \rightarrow 0 \quad (5.11)$$

- the asymptotics (5.10) and (5.11) are uniform on $[-X, X] \times D \times \mathcal{V}_A$;
- the asymptotics (5.10) and (5.11) can be differentiated once in x .

5.5.2

Let us formulate the Theorem WKB on a compact domain.

Theorem 3. [6, 9]

We assume that V satisfies (H_V) and that W satisfies $(H_{W,r})$. Fix $X > 1$ and $E_0 \in \mathbb{C}$. Let $K \subset S_Y$ be a bounded canonical domain with respect to κ . There exists $\varepsilon_0 > 0$ and a consistent basis $\{f_+(x, \varphi, E, \varepsilon), f_-(x, \varphi, E, \varepsilon)\}$ of solutions of (1.3), having the standard behavior (5.10) et (5.11) in K .

For any fixed $x \in \mathbb{R}$, the functions $\varphi \mapsto f_{\pm}(x, \varphi, E, \varepsilon)$ are analytic in K .

5.6 The main geometric tools of the complex WKB method

In this section, we introduce the main geometric tools of the complex WKB method. To do that, we recall some ideas of [5, 6, 7, 27].

5.6.1 Stokes lines

The definition of the Stokes lines is fairly standard, [5, 6]. The integral $\varphi \mapsto \int^\varphi \kappa(u) du$ has the same branch points as the complex momentum. Let φ_0 be one of them. Consider the curves beginning at φ_0 , and described by the equation

$$\text{Im} \int_{\varphi_0}^{\varphi} (\kappa(\xi) - \kappa(\varphi_0)) d\xi = 0 \quad (5.12)$$

These curves are the *Stokes lines* beginning at φ_0 . According to equation (4.11) and equation (4.12), the Stokes line definition is independent of the choice of the branch of κ .

Assume that $W'(\varphi_0) \neq 0$. Equation (4.9) implies that there are exactly three Stokes lines beginning at φ_0 . The angle between any two of them at this point is equal to $\frac{2\pi}{3}$.

5.7 Lines of Stokes type

We recall that $D \subset S_Y$ is a simply connected domain containing no branch points. Let $\gamma \subset D$ be a smooth curve. We say that γ is a line of Stokes type with respect to κ if, along γ , we have

$$\text{either } \text{Im} \left(\int^\varphi \kappa(u) du \right) = \text{Const} \quad \text{or} \quad \text{Im} \left(\int^\varphi (\kappa(u) - \pi) du \right) = \text{Const}$$

5.8 Pre-canonical lines

Let $\gamma \subset D$ be a vertical curve. We call γ *pre-canonical* if it consists of union of bounded segments of canonical curves and/or lines of Stokes type.

5.9 Some branches of the complex momentum

In this section, we describe different branches of κ near the branch points described in 2.4. The geometrical configuration is similar to the one studied in [8].

5.9.1 Different cases

We assume that $(H_{W,r})$, $(H_{W,g})$, and (H_J) are satisfied. To study the geometrical tools of the WKB complex method, one needs to specify the properties of $\text{Im } \kappa$ and $\text{Re } \kappa$. We know that $\kappa(\varphi_r^\pm) \equiv 0[\pi]$ (see section 4). We consider two cases: either $\kappa(\varphi_r^\pm) \equiv 0[2\pi]$ or $\kappa(\varphi_r^\pm) \equiv \pi[2\pi]$.

We define S_- the open domain delimited by the real line at the bottom and by Σ_+ to the right:

$$S_- = \{\varphi - r ; \varphi \in \Sigma_+^*, r \in \mathbb{R}_+^*\} \cap S_Y \quad (5.13)$$

Similarly, we define S_+ the open domain delimited by the real line at the bottom and by Σ_+ to the left:

$$S_+ = \{\varphi + r ; \varphi \in \Sigma_+^*, r \in \mathbb{R}_+^*\} \cap S_Y \quad (5.14)$$

The domains S_+ and S_- are shown in figure 6.
We prove the following result.

Lemma 5.2. *There exists a branch κ_i of the complex momentum such that*

1. $\text{Im } \kappa_i(\varphi) > 0$ for $\varphi \in S_-$, $\kappa_i(\varphi_r^- + i0) = 0$ and $\kappa_i(\varphi_i - 0) = \pi$,
ou
2. $\text{Im } \kappa_i(\varphi) < 0$ for $\varphi \in S_-$, $\kappa_i(\varphi_r^- + i0) = \pi$ and $\kappa_i(\varphi_i - 0) = 0$.

Proof

- First, we specify the sign of $\text{Im } \kappa_i$. The set $(E - W)(\mathbb{R} - [\varphi_r^-, \varphi_r^+])$ belongs to a gap G . We define $\Lambda_- = (E - W)(S_-)$. We prove that Λ_- is a connected domain which intersects with \mathbb{R} only in the gap G . According to assumption $(H_{W,g})$ (2.2.3), there exists a sequence of vertical curves $\tilde{\Sigma}_k$ such that:

$$\Lambda_- \cap \mathbb{R} = (E - W)((-\infty, \varphi_r^-] \cup [\varphi_r^+, +\infty)) \cup (E - W)(\tilde{\Sigma}_k^+).$$

$(E - W)(\tilde{\Sigma}_k^+)$ is a connected domain of \mathbb{R} ; it contains at least a point of G and does not intersect with $\partial\sigma(H_0)$. Consequently, $(E - W)(\tilde{\Sigma}_k^+)$ belongs to G and:

$$\Lambda_- \cap \mathbb{R} = G.$$

We fix on Λ_- a continuous branch of the quasi momentum k . The sign of $\text{Im } k$ does not change since $(E - W)(S_-)$ does not intersect with $\sigma(H_0)$. If we define $\kappa_i(\varphi) = k(E - W(\varphi))$, $\text{Im } \kappa_i$ we can assume that $\text{Im } \kappa_i > 0$ on S_- .

- Now, we consider $\text{Re } \kappa_i$. According to section 4, we can choose the branch κ_i such that $\kappa_i(\varphi_r^- + i0) \in \{0, \pi\}$. First, we study the case $\kappa_i(\varphi_r^-) = 0$; this assumption implies two possibilities.
 1. The point 0 is a minimum for W and the band B in (H_J) is in the form $[E_{4l+1}, E_{4l+2}]$. The points E_r and E_i satisfy $E_r = E_{4l+1}$ and $E_i = E_{4l+2}$. There exists a neighborhood V of $[\varphi_r^-, 0] \cup \sigma$ such that $(E - W)(S_- \cap V) \subset \mathbb{C}_+ \setminus \mathbb{R}$. Actually, in the neighborhood of 0, we have $\text{Im } (E - W(\varphi)) \geq 0$; according to $(H_{W,g})$, there exists a neighborhood V of $[\varphi_r^-, 0] \cup \sigma$ such that $(E - W)(S_- \cap V)$ does not intersect \mathbb{R} . By continuity of the mapping $\varphi \mapsto \text{Im } (E - W(\varphi))$, the sign of $\text{Im } (E - W(\varphi))$ remains positive on $S_- \cap V$. There exists a branch k of the quasi-momentum such that

$$\text{Im } k(\mathcal{E}) > 0 \text{ for } \text{Im } \mathcal{E} > 0 \text{ and } k(E_n + i0) = 0, k(E_p + i0) = \pi.$$

We define $\kappa_i(\varphi) = k(E - W(\varphi))$.

2. The point 0 is a maximum for W and the band B is in the form $[E_{4l+3}, E_{4l+4}]$; the points E_r and E_i satisfy $E_r = E_{4l+4}$ and $E_i = E_{4l+3}$. Let k be the branch of the quasi-momentum such that $\text{Im } k(\mathcal{E}) > 0$ for $\text{Im } \mathcal{E} < 0$, and $k(E_n) = 0$; then $k(E_p) = \pi$. We define $\kappa_i(\varphi) = k(E - W(\varphi))$.

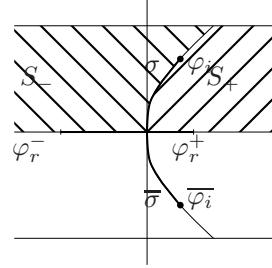


Figure 6: The domains S_- and S_+

The case $\kappa_i(\varphi_r^-) = \pi$ is similar.

This completes the proof of the lemma. \diamond

For the sake of clarity, for all the proofs, we shall consider the case:

$$\kappa_i(\varphi_r^- + i0) = 0 \text{ et } \kappa_i(\varphi_i - 0) = \pi \quad (5.15)$$

The arguments in the second case are similar and we will not give the details.

5.9.2 Other branches of the complex momentum

The properties of the complex momentum near the branch points φ_i , $\overline{\varphi_i}$, φ_r^\pm are determined by the behavior of k near E_r and E_i .

Now, we describe other branches of κ , which are obtained from the branch κ_i (Lemma 5.2) by analytic continuation. We consider the case (5.15).

- We denote by κ_g the continuation of κ_i to the domain $\{\operatorname{Re}(\varphi) < \operatorname{Re}(\varphi_r^-)\}$. κ_g satisfies

$$\begin{aligned} \operatorname{Im}(\kappa_g(\varphi)) &> 0 \text{ for } \{\operatorname{Re}(\varphi) < \operatorname{Re}(\varphi_r^-)\} \\ \operatorname{Re}(\kappa_g)(\varphi) &\rightarrow 0 \text{ as } \operatorname{Re}(\varphi) \rightarrow -\infty. \end{aligned}$$

κ_g is the continuation of κ_i through $(-\infty, \varphi_r^-]$.

- We consider the strip S_Y cut along $(\Sigma \setminus \sigma) \cup (\overline{\Sigma} \setminus \overline{\sigma}) \cup (-\infty, \varphi_r^-) \cup (\varphi_r^+, +\infty)$. We always denote by κ_i the continuation of κ_i through C .
- On $\{\operatorname{Re}(\varphi) > \operatorname{Re}(\varphi_r^+)\}$, we fix a continuous branch κ_d with the conditions:

$$\begin{aligned} \operatorname{Im}(\kappa_d(\varphi)) &> 0 \text{ for } \{\operatorname{Re}(\varphi) > \operatorname{Re}(\varphi_r^+)\} \\ \operatorname{Re}(\kappa_d)(\varphi) &\rightarrow 0 \text{ as } \operatorname{Re}(\varphi) \rightarrow +\infty. \end{aligned}$$

κ_d is the continuation of κ_i through $\overline{S_+}$.

Here, we describe the behavior of the different branches of κ .

$$\forall \varphi \in S_-, \quad \kappa_g(\varphi) = \kappa_i(\varphi) \quad ; \quad \forall \varphi \in \overline{S_-}, \quad \kappa_g(\varphi) = -\kappa_i(\varphi). \quad (5.16)$$

$$\forall \varphi \in S_+, \quad \kappa_d(\varphi) = -\kappa_i(\varphi) \quad ; \quad \forall \varphi \in \overline{S_+}, \quad \kappa_d(\varphi) = \kappa_i(\varphi). \quad (5.17)$$

5.10 Stokes lines

This section is devoted to the description of the Stokes lines under assumptions $(H_{W,g})$ and (H_J) . We describe the Stokes lines beginning at φ_r^- , φ_r^+ , φ_i and $\overline{\varphi_i}$. Since W is real on the real line, the set of the

Stokes lines is symmetric with respect to the real line.

First, κ_i is real on the interval $[\varphi_r^-, \varphi_r^+] \subset \mathbb{R}$; therefore, $[\varphi_r^-, \varphi_r^+]$ is a part of a Stokes line beginning at φ_r^- . The two other Stokes lines beginning at φ_r^- are symmetric with respect to the real line. We denote by b the Stokes line going upward and by \bar{b} its symmetric. Similarly, we denote by a and \bar{a} the two other Stokes lines beginning at φ_r^+ ; a goes upwards.

Consider the Stokes lines beginning at φ_i . The angles between the Stokes lines at this point are equal to $2\pi/3$. So, one of the Stokes lines is situated between Σ and $e^{\frac{2i\pi}{3}}\Sigma$. It is locally going to the right of Σ ; we denote by d this line. Similarly, we denote by e the Stokes line between Σ and $e^{-\frac{2i\pi}{3}}\Sigma$. Finally, we denote by c the third Stokes line beginning at φ_i ; c is going upwards.

By symmetry, we denote by \bar{c} , \bar{d} and \bar{e} the Stokes lines beginning at $\overline{\varphi_i}$.

We describe the behavior of a , b , c , d and e in the strip S_Y . We have represented these lines in figure 7. In this figure, we have precised the values of κ in the branch points.

Lemma 5.3. *We assume that V , W and J satisfy (H_V) , (H_W) and (H_J) . Then, the Stokes lines described in figure 7 have the following properties:*

1. *a stays vertical; it intersects $\{\text{Im } (\varphi) = Y\}$.*
2. *b stays vertical; it intersects $\{\text{Im } (\varphi) = Y\}$.*
3. *d intersects a above φ_r^+ ; the segment between φ_i and this intersection with a is vertical.*
4. *e intersects b above φ_r^- ; the segment between φ_i and this intersection with b is vertical.*
5. *c stays vertical; it intersects $\{\text{Im } (\varphi) = Y\}$ and does not intersect σ .*
6. *a and c do not intersect one another in the strip S_Y .*
7. *b and c do not intersect one another in the strip S_Y .*

Proof First, we note that a Stokes line can become horizontal only at a point where $\text{Im } \kappa = 0$, i.e. at a point of the pre-image of a spectral band. Besides, a Stokes line beginning at φ_1^\pm (respectively at φ_2 or $\overline{\varphi_2}$) is locally orthogonal to $i\kappa(\varphi)$ (respectively $i(\pi - \kappa(\varphi))$).

We first prove 1). According (H_J) , the pre-image of the spectrum is $[\varphi_r^-, \varphi_r^+] \cup \sigma$. So, a becomes horizontal only if it intersects σ . Let us prove by contradiction that it is impossible. Let us assume that a intersects σ in φ_a , then:

$$\begin{aligned} \text{Im } \int_{\varphi_r^+}^{\varphi_a} \kappa(u) du &= 0 = \text{Im } \int_{0, \text{ along } \sigma}^{\varphi_a} \kappa(u) du \\ &= \int_0^{\varphi_a} (\text{Re } \kappa(u)) d(\text{Im } (u)) \leq -k_1(E - W_-) \text{Im } \varphi_a < 0 \end{aligned}$$

which is impossible. Therefore, a stays vertical. Moreover, as $\varphi \rightarrow \infty$, $\varphi \in S_Y$, $\text{Im } (i\kappa) \rightarrow 0$. Thus, a admits a vertical asymptote and intersects $\{\text{Im } (\varphi) = Y\}$.

Similarly, we prove 2).

To prove 3), we consider the Stokes line d . If a and d do not intersect one another, then d intersect either σ or $[0, \varphi_r^+]$. In this case, we denote by φ_d the intersection between d and σ and we have:

$$\text{Im } \int_{\varphi_d}^{\varphi_i} (\kappa(u) - \pi) du = 0 = \int_{\varphi_d}^{\varphi_i} \text{Re } (\kappa(u) - \pi) d(\text{Im } (u)) < 0$$

Consequently, d and a do not intersect one another. Before its intersection with a , d does not intersect the pre-image of a spectral band and it stays vertical. We prove similarly the properties of e .

We prove now 5). c is going upwards. c does not intersect the pre-image of a spectral band in $\{\text{Im } \varphi \in]\text{Im } \varphi_i, Y[\}$ and c stays vertical.

We prove 6) by contradiction. Let us assume that there is $\varphi_a \in a \cap c$. Then, we compute:

$$\text{Im } \int_0^{\varphi_a} \kappa(u) du = 0 = \text{Im } \int_{\sigma} \kappa(u) du + \text{Im } \int_{\varphi_i}^{\varphi_a} \kappa(u) du$$

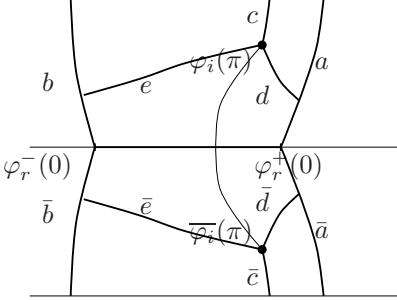


Figure 7: Stokes lines

First, $\text{Im} \int_{\varphi_i}^{\varphi_a} \kappa(u) du = \pi \text{Im} (\varphi_a - \varphi_i) > 0$ and $\text{Im} \int_{\sigma} \kappa(u) du = \int_{\sigma} \text{Re} \kappa(u) d(\text{Im} u) > 0$, which is impossible. So, a and c do not intersect one another in S_Y . \diamond
In the following, we choose $\tilde{Y} \in \sup_{E \in J} \text{Im} \varphi_i(E), Y$.

5.11 Construction of a consistent basis with standard behavior in the neighborhood of the cross

In this section, we begin with constructing a canonical line near the cross. To do that, we follow the methods developed in [9].

5.11.1 General constructions

We first recall some general geometric tools presented in [9], section 4.1.

- We first introduce the idea of enclosing canonical domain.

Definition 5.5. Let $\gamma \subset D$ be a line canonical with respect to κ . Denote its ends by ξ_1 and ξ_2 . Let a domain $K \subset D$ be a canonical domain corresponding to the triple κ, ξ_1 and ξ_2 . If $\gamma \subset K$, then K is called a canonical domain enclosing γ .

We have the following property:

Lemma 5.4. [8]

One can always construct a canonical domain enclosing any given compact canonical curve located in an arbitrarily small neighborhood of that curve.

Such canonical domains, whose existence is established using this lemma are called *local*.

- To construct a canonical domain, we need a canonical line to start with. To construct such a line, we first build pre-canonical lines made of some “elementary” curves. Let $\gamma \subset D$ be a vertical curve. We call γ *pre-canonical* if it is a finite union of bounded segments of canonical lines and/or lines of Stokes type. The interest of pre-canonical curves is the following:

Lemma 5.5. [8]

Let γ be a pre-canonical curve. Denote the ends of γ by ξ_a and x_{i_b} . Fix $V \subset D$, a neighborhood of γ and $V_a \subset D$ a neighborhood of ξ_a . Then, there exists a canonical line $\tilde{\gamma}$ connecting the point ξ_b to some point in V_a .

5.11.2 Constructing a canonical line near the cross

Here, we mimic the construction of [9], section 4.2. We assume that assumptions (H_V) , $(H_{W,r})$, $(H_{W,g})$ and (H'_J) are satisfied. We now explain the construction of a canonical line going from $\{\text{Im} \xi = -Y\}$ to

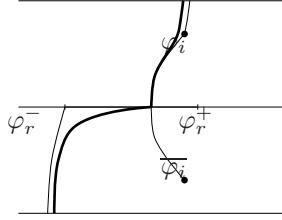


Figure 8: A canonical curve

$\{\text{Im } \xi = Y\}$. First, we consider the curve β which is the union of the Stokes line \bar{b} , the segment $[\varphi_r^-, 0]$ of the real line, the closed curve σ_+ and the Stokes line c .

We now construct α a pre-canonical line close to the line β . We prove:

Proposition 2. *Fix $\delta > 0$. In the δ -neighborhood of β , there exists α a pre-canonical line with respect to the branch κ connecting ξ_1 to ξ_2 and having the following properties:*

- at its upper end, $\text{Im } \xi_2 = Y$,
- at its lower end, $\text{Im } \xi_1 = -Y$,
- it goes around the branch points of the complex momentum as the curve shown in figure 8;
- it contains a canonical line which stays in S_- , goes downward from a point in S_- to the curve σ and then continues along this curve until it intersects the real line.

Proof The proof of this Proposition is completely similar to the proof of Proposition 4.2 in [9]. It consists in breaking down α in “elementary” segments. We do not give the details. \diamond
An immediate consequence of Proposition 2 is the following result:

Proposition 3. *In arbitrarily small neighborhood of the pre-canonical line α , there exists a canonical line γ which has all the properties of the line α listed in Proposition 2.*

5.11.3 Some continuation tools

In this section, we recall some continuation tools; these tools are developed in [9].

1. Now, we present the continuation lemma on compact domains. We recall that q is defined in (5.8).

Lemma 5.6. [6] *Let $\varphi_-, \varphi_+, \varphi_0$ be fixed points such that*

- $\text{Im } \varphi_- = \text{Im } \varphi_+$;
- there is no branch point of $\varphi \mapsto \kappa(\varphi)$ on the interval $[\varphi_-, \varphi_+]$;
- $\varphi_0 \in (\varphi_-, \varphi_+)$, $q(\varphi_0) \neq 0$.

Fix a continuous branch of κ on $[\varphi_-, \varphi_+]$. Let $f(x, \varphi, E, \varepsilon)$, $f_{\pm}(x, \varphi, E, \varepsilon)$ be solutions of (1.3) for $\varphi \in [\varphi_-, \varphi_+]$ and $x \in [-X, X]$ satisfying (1.5) and such that:

- (a) $f(x, \varphi, E, \varepsilon) = e^{\frac{i}{\varepsilon} \int_{\varphi_0}^{\varphi} \kappa(u) du} (\psi_+(x, \varphi, E) + o(1))$ pour $\varphi \in [\varphi_-, \varphi_0]$ for $\varphi \in [\varphi_-, \varphi_0]$ when $\varepsilon \rightarrow 0$ and the asymptotic is differentiable in x ;
- (b) $f_{\pm}(x, \varphi, E, \varepsilon) = e^{\pm \frac{i}{\varepsilon} \int_{\varphi_0}^{\varphi} \kappa(u) du} (\psi_{\pm}(x, \varphi, E) + o(1))$ for $\varphi \in [\varphi_-, \varphi_+]$ when $\varepsilon \rightarrow 0$, and the asymptotic is differentiable in x .

Here, ψ_{\pm} are canonical Bloch solutions associated to the complex momentum κ . Then,

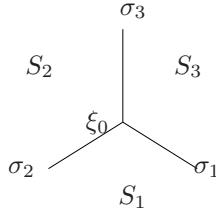


Figure 9: Stokes lines and Stokes Lemma

- if $\operatorname{Im}(\kappa(\varphi)) > 0$ for all $\varphi \in [\varphi_-, \varphi_+]$, there exists $C > 0$ such that, for $\varepsilon > 0$ small enough,

$$\left| \frac{df}{dx}(x, \varphi, E, \varepsilon) \right| + |f(x, \varphi, E, \varepsilon)| \leq C e^{\frac{1}{\varepsilon} \int_{\varphi_0}^{\varphi} |\kappa(u)| du} |\operatorname{Im} \kappa(u)| du, \quad \varphi \in [\varphi_0, \varphi_+]; \quad (5.18)$$

- if $\operatorname{Im}(\kappa(\varphi)) < 0$ for all $\varphi \in [\varphi_-, \varphi_+]$, then

$$f(x, \varphi, E, \varepsilon) = e^{\frac{i}{\varepsilon} \int_{\varphi_0}^{\varphi} \kappa(u) du} (\psi_+(x, \varphi, E) + o(1)), \quad \varphi \in [\varphi_0, \varphi_+], \quad (5.19)$$

and the asymptotic is differentiable in x .

Intuitively, this lemma means that a function f has the standard behavior along a horizontal line as long as the leading term of its asymptotics is growing along that line. For analogous results with real WKB method, we refer to [26].

2. The estimate we obtained in Lemma 5.6 can be far from optimal. The Adjacent Canonical Domain Principle gives a more precise result:

Proposition 4. [7] Assume that a solution f has standard behavior in either the left hand side or the right hand side of a constant neighborhood of a vertical curve γ . Assume that γ is canonical with respect to some branch of the complex momentum. Then f has standard behavior in any bounded canonical domain enclosing γ .

3. The last tool we shall need in the sequel is the Stokes Lemma.

Notations and assumptions:

Assume that ξ_0 is a branch point of the complex momentum such that $W'(\xi_0) \neq 0$. There are three Stokes lines beginning at ξ_0 . The angles between them at ξ_0 are equal to $2\pi/3$. We denote these lines by σ_1 , σ_2 and σ_3 , so that σ_1 is vertical at ξ_0 . Let V be a neighborhood of ξ_0 ; assume that V is so small that σ_1 , σ_2 and σ_3 divide it into three sectors. We denote them by S_1 , S_2 and S_3 so that S_1 be situated between σ_1 and σ_2 , and the sector S_2 be between σ_2 and σ_3 (see figure 9).

We recall now the result:

Lemma 5.7. [9]. Let V be sufficiently small. Let f be a solution that has standard behavior $f = e^{\frac{i}{\varepsilon} \int^{\varphi} \kappa(u) du} (\psi_+(x, \varphi) + o(1))$ inside the sector $S_1 \cup \sigma_2 \cup S_2$ of V . Moreover, assume that, in S_1 near σ_1 , one has $\operatorname{Im} \kappa > 0$ if S_1 is to the left of σ_1 and $\operatorname{Im} \kappa < 0$ otherwise. Then, f has standard behavior $f = e^{\frac{i}{\varepsilon} \int^{\varphi} \kappa(u) du} (\psi_+(x, \varphi) + o(1))$ inside $V \setminus \sigma_1$, the asymptotics being obtained by analytic continuation from $S_1 \cup \sigma_2 \cup S_2$ to $V \setminus \sigma_1$.

5.11.4 Construction of a basis with standard asymptotic behavior near the cross

We prove the existence of a consistent basis with standard asymptotic behavior near the canonical line α . Let α be the curve described in Proposition 3. According to Lemma 5.4, we can construct a local

canonical domain K_i enclosing α .

Proposition 5. *Assume that (H_V) , $(H_{W,r})$, $(H_{W,g})$ and (H_J) are satisfied. Fix $E_0 \in J$, $X > 1$ and $\tilde{Y} \in]0, \tilde{Y}[$. Then, there exist a complex neighborhood \mathcal{U}_0 of E_0 , a real number $\varepsilon_0 > 0$ and a function f_i satisfying the following properties:*

- The function $(x, \varphi, E, \varepsilon) \mapsto f_i(x, \varphi, E, \varepsilon)$ is defined on $\mathbb{R} \times S_{\tilde{Y}} \times \mathcal{U}_0 \times]0, \varepsilon_0[$.
- For any $x \in \mathbb{R}$, for any $\varepsilon \in]0, \varepsilon_0[$, the function $((\varphi, E) \mapsto f_i(x, \varphi, E, \varepsilon))$ is analytic on $S_{\tilde{Y}} \times \mathcal{U}_0$.
- For $(x, \varphi, E) \in [-X, X] \times K_i \times \mathcal{U}_0$, the function f_i has the asymptotic behavior:

$$f_i(x, \varphi, E, \varepsilon) = e^{\frac{i}{\varepsilon} \int_0^\varphi \kappa(u) du} (\psi_{\pm}(x, \varphi, E) + o(1)), \quad \varepsilon \rightarrow 0. \quad (5.20)$$

- The asymptotics (5.20) are uniform in $(x, \varphi, E) \in [-X, X] \times K_i \times \mathcal{U}_0$.
- The asymptotics (5.20) can be differentiated once in x .
- There exists a real number $\sigma_i \in \{-1, 1\}$ such that the function f_i satisfies the relation:

$$w(f_i, f_i^*) = w(f_i(\cdot, \varphi, E, \varepsilon), f_i(\cdot, \bar{\varphi}, \bar{E}, \varepsilon)) = \sigma_i(k'_i w_i)(E - W(0))$$

The end of this section is devoted to the proof of Proposition 5. This Proposition mainly follows from Theorem 3.

5.11.5 Existence of f_i

The domain K_i is a local canonical domain. According to Theorem 3, we can build a function f_i such that, on K_i , f_i has the following asymptotic behavior:

$$f_i \sim e^{\frac{i}{\varepsilon} \int^\varphi \kappa_i} \psi_+.$$

Let us normalize f_i in 0.

5.11.6 Computation of the Wronskian $w(f_i, f_i^*)$

To finish the proof of Proposition 5, it remains to compute $w(f_i, f_i^*)$.

Let R_+ be a small enough rectangle to the left of α_+ , so that $R_+ \subset K_i \cap S_{\tilde{Y}}$. We define $R = R_+ \cup R_-$; we study the behavior of f_i and $(f_i)^*$ in R .

- First, by construction, in R_+ , the function f_i satisfies:

$$f_i \sim e^{\frac{i}{\varepsilon} \int_0^\varphi \kappa_i} \psi_+^i.$$

- To the right of α_- , the function f_i satisfies

$$f_i \sim e^{\frac{i}{\varepsilon} \int_0^\varphi \kappa_i} \psi_+^i,$$

with $\text{Im } \kappa_i < 0$. According to Lemma 5.6, we know that, in \tilde{S}_- , the function f_i admits the asymptotic behavior:

$$f_i \sim e^{\frac{i}{\varepsilon} \int_0^\varphi \kappa_i} \psi_+^i.$$

- Thus, the function f_i has the standard asymptotic behavior in R .
- Now, we study the behavior of f_i^* . To do that, we start with describing the main objects related to κ_i in R . Let k_i be the branch of the quasi-moment of (4.1), analytically continued through $[E_r, E_i]$ and satisfying:

$$k_i(E_r) = 0 \quad \text{and} \quad k_i(E_i) = \pi$$

k_i is real on $[E_r, E_i]$. Therefore, k_i satisfies:

$$k_i(\bar{E}) = \overline{k_i(E)}.$$

The branch κ_i satisfies $\kappa_i(\varphi) = k_i(E - W(\varphi))$. The associated canonical Bloch solutions Ψ_\pm^i are such that:

$$\overline{\Psi_+^i(x, \bar{\varphi})} = \Psi_-^i(x, \varphi).$$

Therefore, we have in R :

$$\kappa_i^*(\varphi) = \kappa_i(\varphi) \quad (\Psi_+^i)^*(\varphi) = \Psi_-^i(\varphi) \quad (\omega_+^i)^*(\varphi) = \omega_-^i(\varphi) \quad \forall \varphi \in R. \quad (5.21)$$

Besides, since k'_i is real on the band, there exists a real number $\sigma_i \in \{-1, 1\}$ such that:

$$q_i^*(\varphi) = \sigma_i q_i(\varphi). \quad (5.22)$$

We shall precise this coefficient in section 8.1.2.

We compute:

$$w(f_+^i(\cdot, \varphi, E, \varepsilon), (f_+^i)^*(\cdot, \varphi, E, \varepsilon)) = q_i(0)q_i^*(0)w(\Psi_+^i(\cdot, 0), \Psi_-^i(\cdot, 0))g(\varphi, E, \varepsilon).$$

Since $\overline{w(\Psi_+^i(\cdot, 0), \Psi_-^i(\cdot, 0))} = -w(\Psi_+^i(\cdot, 0), \Psi_-^i(\cdot, 0))$, the term $g(\varphi, E, \varepsilon)$ satisfies:

$$g^*(\varphi, E, \varepsilon) = \overline{g(\bar{\varphi}, \bar{E}, \varepsilon)} = g(\varphi, E, \varepsilon),$$

$$g(\varphi, E, \varepsilon) = [1 + o(1)].$$

Since the Wronskian is analytic and ε -periodic, this asymptotic is valid in $S_{\bar{Y}}$.

Since $g^* = g$ and $g = [1 + o(1)]$, there exists an analytic function $(\varphi, E) \mapsto h(\varphi, E, \varepsilon)$ on $S_{\bar{Y}} \times \mathcal{U}$ such that:

- $g(\varphi, E, \varepsilon) = h(\varphi, E, \varepsilon)h^*(\varphi, E, \varepsilon)$,
- $h(\varphi, E, \varepsilon) = [1 + o(1)]$.

We slightly deform f_i , i.e., we replace f_i by $\frac{f_i}{h(\varphi, E, \varepsilon)}$; the basis $\{f_i, f_i^*\}$ is consistent.

This ends the proof of Proposition 5.

6 Consistent Jost solutions of (1.3)

This section is devoted to the proof of the following result.

Proposition 6. *We assume that (H_V) , $(H_{W,r})$ and (H_0^0) are satisfied. Fix $X > 1$ and $\lambda > 1$. Then, there exist a complex neighborhood $\mathcal{V} = \bar{\mathcal{V}}$ of J , a real $\varepsilon_0 > 0$, a constant $C > 0$, two complex numbers m_g , m_d and two functions $(x, \varphi, E, \varepsilon) \mapsto h_-^g(x, \varphi, E, \varepsilon)$, $(x, \varphi, E, \varepsilon) \mapsto h_+^d(x, \varphi, E, \varepsilon)$ such that, if we define*

$$B_\varepsilon^g = \left\{ \varphi \in S_Y ; \operatorname{Re} \varphi < -C\varepsilon^{-\frac{\lambda}{s-1}} \right\} \text{ et } B_\varepsilon^d = \left\{ \varphi \in S_Y ; \operatorname{Re} \varphi > C\varepsilon^{-\frac{\lambda}{s-1}} \right\},$$

then

- The functions $(x, \varphi, E, \varepsilon) \mapsto h_-^g(x, \varphi, E, \varepsilon)$ and $(x, \varphi, E, \varepsilon) \mapsto h_+^d(x, \varphi, E, \varepsilon)$ are clearly defined and consistent on $\mathbb{R} \times S_Y \times \mathcal{V} \times]0, \varepsilon_0[$.
- For any $x \in [-X, X]$ and $\varepsilon \in]0, \varepsilon_0[$, $(\varphi, E) \mapsto h_-^g(x, \varphi, E, \varepsilon)$ and $(\varphi, E) \mapsto h_+^d(x, \varphi, E, \varepsilon)$ are analytic on $S_Y \times \mathcal{V}$.
- The function $x \mapsto h_-^g(x, \varphi, E, \varepsilon)$ (resp. $x \mapsto h_+^d(x, \varphi, E, \varepsilon)$) is a basis of \mathcal{J}_- (resp. \mathcal{J}_+).
- The functions h_-^g and h_+^d have the following asymptotic behavior:

$$h_-^g(x, \varphi, E, \varepsilon) = e^{\frac{-i}{\varepsilon} \int_{m_g}^\varphi \kappa(u) du} \psi_-(x, \varphi, E)(1 + R_g(x, \varphi, E, \varepsilon)), \quad (6.1)$$

and

$$h_+^d(x, \varphi, E, \varepsilon) = e^{\frac{i}{\varepsilon} \int_{m_d}^\varphi \kappa(u) du} \psi_+(x, \varphi, E)(1 + R_d(x, \varphi, E, \varepsilon)), \quad (6.2)$$

where

– R_g and R_d satisfy:

$$\exists M > 0, \quad \forall \varepsilon \in]0, \varepsilon_0[, \quad \forall x \in [-X, X], \quad \forall E \in \mathcal{V}, \quad \forall \varphi \in B_\varepsilon^g, \quad |R_g(x, \varphi, E, \varepsilon)| \leq \frac{M}{\varepsilon |Re \varphi|^{s-1}},$$

$$\exists M > 0, \quad \forall \varepsilon \in]0, \varepsilon_0[, \quad \forall x \in [-X, X], \quad \forall E \in \mathcal{V}, \quad \forall \varphi \in B_\varepsilon^d, \quad |R_d(x, \varphi, E, \varepsilon)| \leq \frac{M}{\varepsilon |Re \varphi|^{s-1}}.$$

- The functions ψ_+ and ψ_- are the Bloch canonical solutions of the periodic equation (3.1) defined in section 5.2.
- The asymptotics (6.1) and (6.2) may be differentiated once in x .
- There exist two real numbers $\sigma_g \in \{-1, 1\}$, $\sigma_d \in \{-1, 1\}$, an integer p and two functions $E \mapsto \alpha_g(E)$ and $E \mapsto \alpha_d(E)$ such that:
 1. For any $\varepsilon \in]0, \varepsilon_0[, x \in \mathbb{R}, E \in \mathcal{V}$, et $\varphi \in S_Y$, we have:

$$\overline{\alpha_g(E)h_-^g(x, \overline{\varphi}, \overline{E}, \varepsilon)} = i\sigma_g e^{-\frac{i}{\varepsilon}2p\pi x} \alpha_g(E)h_-^g(x, \varphi, E, \varepsilon) \quad (6.3)$$

$$\overline{\alpha_d(E)h_+^d(x, \overline{\varphi}, \overline{E}, \varepsilon)} = i\sigma_d e^{\frac{i}{\varepsilon}2p\pi x} \alpha_d(E)h_+^d(x, \varphi, E, \varepsilon) \quad (6.4)$$

2. The functions α_g and α_d are analytic and given by (6.22) and (6.25). They do not vanish on \mathcal{V} .

We shall construct some consistent Jost solutions of (1.3). To do that, we regard equation (1.3) as a perturbation of equation (2.1) with $\mathcal{E} = E$. We adapt the construction of Jost functions developed in [10, 17]. Precisely, we look for solutions of (2.1) in the form :

$$F_-^g = e^{-ik(E)\varphi/\varepsilon} \psi_-^0(x, E)(1 + o(1)), \quad x \rightarrow -\infty,$$

$$F_+^d = e^{ik(E)\varphi/\varepsilon} \psi_+^0(x, E)(1 + o(1)), \quad x \rightarrow +\infty.$$

Since the functions $(x, \varphi, E, \varepsilon) \mapsto e^{\pm ik(E)\varphi/\varepsilon} \psi_\pm(x, E)$ are consistent, they allow us to construct a consistent resolvent for the periodic equation. Using this property and the fact that equation (1.3) is invariant by the consistency transformation $(x, \varphi) \mapsto (x - 1, \varphi + \varepsilon)$, we obtain the consistency of the Jost functions.

6.1 Construction of the Jost functions

We start with constructing F_-^g . The construction of F_+^d is similar. Since the parameter E lies in the neighborhood of a gap, $\text{Im } k(E)$ is non zero; the function F_-^g is therefore exponentially decreasing and goes to zero as x goes to $-\infty$. Such a solution is called recessive.

6.1.1

On a small enough complex neighborhood of J , $\mathcal{V} = \overline{\mathcal{V}}$, one can fix a determination k of the quasi-momentum such that:

$$\text{Im } k(E) \geq \beta > 0, \quad \forall E \in \mathcal{V}.$$

Fix m_g in S_Y such that:

- The point m_g is not a branch point of κ .
- It satisfies $\text{Im } m_g > 0$, $k'_E(m_g) \neq 0$.
- The domain $\{\varphi \in S_Y ; \text{Re } (\varphi - m_g) < 0 \text{ and } \text{Im } (\varphi - m_g) > 0\}$ does not contain any branch point of κ .

We define $E_g = E - W(m_g)$. We denote by ψ_\pm^0 the analytic Bloch solutions of equation (2.1) normalized at the point E_g ($k'(E_g) \neq 0$). These solutions are constructed in Lemma 4.1.

$$\psi_\pm^0(x, E) = e^{\pm ik(E)x} p_\pm^0(x, E) \quad \text{with} \quad p_\pm^0(x+1, E) = p_\pm^0(x, E).$$

We define :

$$\widetilde{\psi}_\pm(x, \varphi, E, \varepsilon) = e^{\pm ik(E)(x+\frac{\varphi}{\varepsilon})} p_\pm^0(x, E) = e^{\pm ik(E)\frac{\varphi}{\varepsilon}} \psi_\pm^0(x, E).$$

We consider the resolvent R of H_0 :

$$(Rg)(x) = - \int_{-\infty}^x \frac{\psi_+^0(x, E)\psi_-^0(x', E) - \psi_+^0(x', E)\psi_-^0(x, E)}{(k'w_0)(E_g)} g(x') dx'$$

6.2

Since $\widetilde{\psi}_-$ goes to zero as x goes to $-\infty$, we look for a recessive consistent solution \tilde{f} of (1.3) in the form :

$$\tilde{f}(x, \varphi, E, \varepsilon) = \widetilde{\psi}_-(x, \varphi, E, \varepsilon) + R[W(\varepsilon x + \varphi)\tilde{f}(x, \varphi, E, \varepsilon)]. \quad (6.5)$$

We define $\tilde{f}(x, \varphi, E, \varepsilon) = e^{-ik(E)(x+\frac{\varphi}{\varepsilon})} f(x, \varphi, E, \varepsilon)$; equation (6.5) is transformed into:

$$f(x, \varphi, E, \varepsilon) = p_-^0(x, E) + \int_{-\infty}^x A(x, x', E) W(\varepsilon x' + \varphi) f(x', \varphi, E, \varepsilon) dx' \quad (6.6)$$

where the function A satisfies:

$$A(x, x', E) = \frac{e^{2ik(E)(x-x')} p_+^0(x, E) p_-^0(x', E) - p_+^0(x', E) p_-^0(x, E)}{(k'w_0)(E_g)}. \quad (6.7)$$

Since $\text{Im } k(E) \geq \beta > 0$ for $E \in \mathcal{V}$, there exists a constant $C > 0$ such that:

$$\forall x > x', \quad \forall E \in \mathcal{V}, \quad |A(x, x', E)| \leq C \quad (6.8)$$

6.2.1

Fix $X_0 \in \mathbb{R}$ and $a > 0$. If I is a real interval, we define:

$$R_I = \{\varphi \in S_Y ; \text{Re } \varphi \in I\}.$$

Let $B((-\infty, X_0] \times R_{[-a, a]})$ the set of bounded functions $\{f : (x, \varphi) \mapsto f(x, \varphi)\}$ on $(-\infty, X_0] \times R_{[-a, a]}$. The set $B((-\infty, X_0] \times R_{[-a, a]})$ equipped with the norm

$$\|f\|_\infty = \sup_{x \in (-\infty, X_0], \text{Re } \varphi \in [-a, a]} |f(x, \varphi)|$$

is a Banach space.

We define the integral operator T_E by:

$$\begin{aligned} T_E : B((-\infty, X_0] \times R_{[-a, a]}) &\rightarrow B((-\infty, X_0] \times R_{[-a, a]}) \\ f &\mapsto F \end{aligned}$$

$$\text{where } F(x, \varphi) = \int_{-\infty}^x A(x, x', E) W(\varepsilon x' + \varphi) f(x', \varphi) dx'. \quad (6.9)$$

The operator T_E is a bounded operator on $B((-\infty, X_0] \times R_{[-a, a]})$ and satisfies the estimate:

$$\forall x \in (-\infty, X_0], \quad \forall \varphi \in R_{[-a, a]}, \quad |T_E(f)(x, \varphi)| \leq C \|f\|_\infty \int_{-\infty}^x |W(\varepsilon x' + \text{Re } (\varphi) + i\text{Im } (\varphi))| dx'.$$

$$\|T_E(f)\|_\infty \leq \frac{M}{\varepsilon} \sup_{x \in (-\infty, X_0], \text{Re } \varphi \in [-a, a]} \frac{1}{|\varepsilon x + \text{Re } (\varphi)|^{s-1}}.$$

6.2.2

Fix $\lambda > 1$. There exists a constant $C > 0$ such that:

$$|X_0| > C\varepsilon^{-\frac{\lambda s}{s-1}} \Rightarrow \|\|T_E\|\| < \varepsilon^{s(\lambda-1)}.$$

We rewrite (6.6) in the form:

$$(1 - T_E)f = p_-^0(x, E) \quad (6.10)$$

The operator is then invertible on $B((-\infty, X_0] \times R_{[-a, a]})$. We define:

$$F_-^g(x, \varphi, E, \varepsilon) = (1 - T_E)^{-1}p_-^0(x, \varphi, E, \varepsilon) \quad (6.11)$$

We now give some properties of F_-^g .

6.3 Properties of F_-^g

6.3.1 Asymptotic behavior in x

Substituting

$$(1 - T_E)^{-1} = 1 - (1 - T_E)^{-1}T_E$$

in equation (6.11), we obtain:

$$F_-^g(x, \varphi, E, \varepsilon) = e^{-ik(E)\varphi/\varepsilon}\psi_-(x, E)(1 + R_g(x, \varphi, E, \varepsilon)), \quad (6.12)$$

with

$$|R_g(x, \varphi, E, \varepsilon)| \leq \frac{M}{\varepsilon|\varepsilon x|^{s-1}},$$

for $x \in (-\infty, X_0]$ and $\varphi \in R_{[-a, a]}$.

The function F_-^g is therefore in the Jost subspace \mathcal{J}_- of equation (1.3).

6.3.2 Study of the consistency

We assume that $a > 1$ and $\varepsilon < 1$. We now prove that the function F_-^g is consistent.

We denote by G the function:

$$G : (x, \varphi, E, \varepsilon) \mapsto G(x, \varphi, E, \varepsilon) = F_-^g(x + 1, \varphi - \varepsilon, E, \varepsilon)$$

G is defined for $x \in (-\infty, X_0 - 1]$ and $\varphi \in R_{[-a+1, a-1]}$. Moreover, the function G belongs to $B((-\infty, X_0 - 1] \times R_{[-a+1, a-1]})$.

We define the operator:

$$\widetilde{T}_E : \begin{array}{ccc} B((-\infty, X_0 - 1] \times R_{[-a+1, a-1]}) & \rightarrow & B((-\infty, X_0 - 1] \times R_{[-a+1, a-1]}) \\ f & \mapsto & F \end{array}$$

$$\text{where } F(x, \varphi) = \int_{-\infty}^x A(x, x', E)W(\varepsilon x' + \varphi)f(x', \varphi)dx'. \quad (6.13)$$

Since $B((-\infty, X_0] \times R_{[-a, a]}) \subset B((-\infty, X_0 - 1] \times R_{[-a+1, a-1]})$ and according to equations (6.9) and (6.13), the operator \widetilde{T}_E is an extension of the operator T_E . Let us denote by \widetilde{F}_-^g the restriction of F_-^g to $(-\infty, X_0 - 1] \times R_{[-a+1, a-1]}$.

We compute in $B((-\infty, X_0 - 1] \times R_{[-a+1, a-1]})$:

$$(\widetilde{T}_E(G))(x, \varphi, E, \varepsilon) = (T_E(F_-^g))(x + 1, \varphi - \varepsilon, E, \varepsilon)$$

This leads to:

$$((1 - \widetilde{T}_E)(G))(x + 1, \varphi - \varepsilon, E, \varepsilon) = ((1 - T_E)(F_-^g))(x + 1, \varphi - \varepsilon, E, \varepsilon) = p_-^0(x + 1, E) = p_-^0(x, E),$$

The functions \widetilde{F}_-^g and G satisfy the relation:

$$((1 - \widetilde{T}_E)(G)) = ((1 - \widetilde{T}_E)(\widetilde{F}_-^g)).$$

For a sufficiently small ε_0 , the operator \widetilde{T}_E satisfies, for any $\varepsilon \in]0, \varepsilon_0[$:

$$\|\widetilde{T}_E\| < \frac{1}{2}.$$

The operator $(1 - \widetilde{T}_E)$ is invertible in $B((-\infty, X_0 - 1] \times R_{[-a+1, a-1]})$ and:

$$\widetilde{F}_-^g = G.$$

For $\varphi \in R_{[-a+1, a-1]}$, the functions F_-^g and G coincide on $(-\infty, X_0 - 1]$; according to the Cauchy-Lipschitz Theorem, they coincide for $x \in \mathbb{R}$. Fix $x \in \mathbb{R}$; F_-^g and G coincide for $\varphi \in R_{[-a+1, a-1]}$. By analyticity, they are equal for $\varphi \in S_Y$.

6.3.3 Asymptotic behavior in φ

We use now the consistency of F_-^g to compute its asymptotics as $\operatorname{Re} \varphi$ goes to $-\infty$. Fix $X > 0$. We study F_-^g for $x \in [-X, X]$. The function F_-^g is consistent, and:

$$\begin{aligned} F_-^g(x, \varphi, E, \varepsilon) &= F_-^g(x + \frac{[\operatorname{Re}(\varphi)]}{\varepsilon}, \varphi - [\operatorname{Re}(\varphi)], E, \varepsilon) \\ &= e^{-ik(E)(x+\varphi/\varepsilon)} p_-^0(x, E) \left(1 + O(\frac{1}{\varepsilon |\varepsilon x + [\operatorname{Re} \varphi]|^{s-1}}) \right). \end{aligned}$$

As a result, there exists a constant C such that:

$$\operatorname{Re} \varphi < -C\varepsilon^{-\frac{\lambda}{s-1}} \Rightarrow F_-^g(x, \varphi, E, \varepsilon) = e^{-ik(E)(x+\varphi/\varepsilon)} p_-^0(x, E) (1 + \widetilde{R}_g(x, \varphi, E, \varepsilon)),$$

where

$$|\widetilde{R}_g(x, \varphi, E, \varepsilon)| \leq \frac{M}{\varepsilon |\operatorname{Re} \varphi|^{s-1}},$$

for $x \in [-X, X]$ and $\operatorname{Re} \varphi < -C\varepsilon^{-\frac{\lambda}{s-1}}$.

We define $B_\varepsilon^g = \{\varphi \in S_Y ; \operatorname{Re} \varphi < -C\varepsilon^{-\frac{\lambda}{s-1}}\}$.

6.4 Renormalization of F_-^g

We now renormalize F_-^g . We define:

$$f_-^g(x, \varphi, E, \varepsilon) = e^{-\frac{i}{\varepsilon} \int_{m_g}^{-\infty} [\kappa(u) - k(E)] du} F_-^g(x, \varphi, E, \varepsilon), \quad (6.14)$$

where the integral $\int_{m_g}^{-\infty} [\kappa(u) - k(E)] du$ is taken in the upper half plane. The function $E \mapsto \int_{m_g}^{-\infty} [\kappa - k(E)]$ is analytic on \mathcal{V} . For $\varphi \in B_\varepsilon^g$, we have:

$$f_-^g(x, \varphi, E, \varepsilon) = e^{-\frac{i}{\varepsilon} \int_{m_g}^{\varphi} [\kappa(u) - k(E)] du} e^{-\frac{i}{\varepsilon} \int_{\varphi}^{-\infty} [\kappa - k(E)] du} e^{-\frac{ik(E)\varphi}{\varepsilon}} \psi_-^0(x, E) (1 + o(1))$$

Since the function ψ_- is analytic and since $W(\varphi) = O(\varepsilon^{\frac{\lambda s}{s-1}})$ for $\varphi \in B_\varepsilon^g$, we get:

$$\forall \varphi \in B_\varepsilon^g, \quad \psi_-(x, \varphi, E) = \psi_-^0(x, E - W(\varphi)) = \psi_-^0(x, E) (1 + o(1)) \quad (6.15)$$

We finally obtain that, for $x \in [-X, X]$ and $\varphi \in B_\varepsilon^g$:

$$f_-^g(x, \varphi, E, \varepsilon) = e^{-\frac{i}{\varepsilon} \int_{m_g}^{\varphi} \kappa(u) du} \psi_-(x, \varphi, E) (1 + o(1))$$

6.4.1 Symmetries

Let γ be a complex path and f be an analytic function on γ . We have:

$$\int_{\gamma} f(z) dz = \overline{\int_{\bar{\gamma}} f^*(z) dz}. \quad (6.16)$$

Since J satisfies (H_J^0) , according to equation (4.12), there exists an integer p such that:

$$k(E) + k^*(E) = 2p\pi. \quad (6.17)$$

We recall that the functions ω_{\pm} associated to κ are defined by equation (5.7). We consider a path $\tilde{\gamma}_g$ such that:

- The path $\tilde{\gamma}_g$ connects $\overline{m_g}$ to m_g and is symmetric with respect to the real axis.
- The path $\tilde{\gamma}_g$ does not contain any branch point of κ and any pole of ω_{\pm} .

We fix a continuous determination q_g of $\sqrt{k'_E}$ on γ_g . According to relation (4.12), we have $(k^*)' = -k'$, which implies that there exists $\sigma_g \in \{-1, 1\}$ such that:

$$q_g^* = i\sigma_g q_g \quad (6.18)$$

The functions $\psi_{\pm}(x, \varphi, E, m_g)$ satisfy the relation:

$$\psi_{\pm}^*(x, \varphi, E, m_g) = i\sigma_g e^{\pm \frac{2ip\pi x}{\varepsilon}} e^{\int_{\tilde{\gamma}_g} \omega_{\pm}^g} \psi_{\pm}^*(x, \varphi, E, m_g). \quad (6.19)$$

Besides, equations (6.16) and (4.17) lead to the following relations:

$$\overline{\int_{\tilde{\gamma}_g} \omega_{+}} = - \int_{\tilde{\gamma}_g} \omega_{+}; \quad \overline{\int_{\tilde{\gamma}_g} \omega_{-}} = - \int_{\tilde{\gamma}_g} \omega_{-}. \quad (6.20)$$

According to $(H_{W,r})$, $W^* = W$. By using (6.7), we compute:

$$A(x, x', E) = \overline{A(x, x', \bar{E})}.$$

The operator T_E satisfies:

$$T_E(f^*) = [T_E(f)]^*.$$

Consequently, according to (6.10) and (6.19), we obtain that, for E in \mathcal{V} , x in \mathbb{R} and φ in B_{ε}^g ,

$$(F_-^g)^*(x, \varphi, E, \varepsilon) = \overline{F_-^g(x, \bar{\varphi}, \bar{E}, \varepsilon)} = i\sigma_g e^{-\frac{i}{\varepsilon}2p\pi x} e^{\int_{\tilde{\gamma}_g} \omega_{-}^g} F_-^g(x, \varphi, E, \varepsilon). \quad (6.21)$$

This leads to:

$$(h_{-}^g)^* = i\sigma_g e^{-\frac{i}{\varepsilon}2p\pi x} \frac{\alpha_g(E)}{\alpha_g^*(E)} h_{-}^g,$$

where

$$\alpha_g(E) = e^{-\frac{i}{2\varepsilon} \left(\int_{\tilde{\gamma}_g} (\kappa(u) - p\pi) du + p\pi(m_g + \overline{m_g}) \right)} e^{\frac{1}{2} \int_{\tilde{\gamma}_g} \omega_{-}^g} \quad (6.22)$$

Similarly, we fix m_d in S_Y such that:

- The point m_d is not a branch point of κ .
- It satisfies $\text{Im } m_d > 0$, $k'_E(m_d) \neq 0$.
- The domain $\{\varphi \in S_Y ; \text{Re } (\varphi - m_d) > 0 \text{ and } \text{Im } (\varphi - m_d) > 0\}$ does not contain any branch point of κ .

We define $E_d = E - W(m_d)$ and we define the function h_+^d by:

$$h_+^d(x, \varphi, E, \varepsilon) = e^{\frac{i}{\varepsilon} \int_{m_d}^{+\infty} [\kappa(u) - k(E)] du + \frac{ik(E)m_d}{\varepsilon}} F_+^d(x, \varphi, E, \varepsilon) \quad (6.23)$$

where the integral $\int_{m_d}^{+\infty} [\kappa(u) - k(E)] du$ is taken in the upper half plane.

We consider the path $\tilde{\gamma}_d$ such that:

- The path $\tilde{\gamma}_d$ connects $\overline{m_d}$ to m_d and is symmetric with respect to the real axis.
- The path $\tilde{\gamma}_d$ does not contain any branch point of κ and any pole of ω_{\pm} .

We fix a continuous branch q_d of $\sqrt{k_E}$ on γ_d . There exists a real number σ_d such that:

$$q_d^* = i\sigma_d q_d \quad (6.24)$$

The function h_+^d satisfies:

$$(h_+^d)^* = i\sigma_d e^{\frac{i}{\varepsilon}2p\pi x} \frac{\alpha_d(E)}{\alpha_d^*(E)} h_+^d,$$

where

$$\alpha_d(E) = e^{\frac{i}{2\varepsilon} \left(\int_{\tilde{\gamma}_d} (\kappa(u) - p\pi) du + p\pi(m_d + \overline{m_d}) \right)} e^{\frac{1}{2} \int_{\tilde{\gamma}_d} \omega_+^d} \quad (6.25)$$

We define *the transmission coefficient*:

$$d(\varphi, E, \varepsilon) = w(\alpha_g h_-^g(\cdot, \varphi, E, \varepsilon), \alpha_d h_+^d(\cdot, \varphi, E, \varepsilon)) \quad (6.26)$$

We immediately deduce from Proposition 1 and Proposition 6 that the eigenvalues of $H_{\varphi, \varepsilon}$ are characterized by:

$$d(\varphi, E, \varepsilon) = 0 \quad (6.27)$$

6.5 Some remarks

6.5.1

The assumption $(H_{W,r})$ is not optimal. Actually, it suffices to assume that W is analytic real in S_Y and that there exists a function $f \in L^1(\mathbb{R})$ such that :

$$\forall x \in \mathbb{R} \quad \sup_{y \in [-Y, Y]} |W(x + iy)| \leq f(x).$$

6.5.2

In equations (6.3) and (6.4), we could have included the numbers $i\sigma_g$ and $i\sigma_d$ into the functions α_g and α_d , but we prefer showing the relations between q_g and q_g^* , q_d and q_d^* .

6.5.3

Note that this construction differs from the constructions of canonical domains in [6]. Indeed, the domains on which we construct these functions depend on ε . We shall extend these asymptotics on a fixed strip in the neighborhood of the real line (section 7).

7 WKB Theorem on non compact domains

In this section, we prove a continuation result on non compact domains of S_Y . This result is a generalization on non compact domains of the method developed in [6] and particularly of Lemma 5.6. We prove that the continuation of asymptotics stay valid on some half-strips $\{\varphi \in S_Y ; |\operatorname{Re} \varphi| > A\}$. To do that, we cover these domains by a countable union of small local overlapping canonical domains, called δ -chain (see section 7.1.5).

This principle follows the recent developments and improvements of the WKB method (see [7]). The idea is to get over the local notion of canonical domain in favor of maximal domains. These domains, constructed as union of local canonical domains are some domains on which a function keeps the standard behavior (see [7]).

7.1 Continuation Theorem on non compact domains

7.1.1 The main result

We shall prove the following result:

Theorem 4. *Continuation Theorem on non compact domains.*

Fix $\tilde{Y} \in]0, Y[$. Assume that V satisfies (H_V) , that W satisfies $(H_{W,r})$ and that J satisfies (H_J^0) . Then, there exist a real $\varepsilon_0 > 0$, a complex neighborhood \mathcal{V} of J and two real numbers A_g and A_d such that, if f has the following properties:

- The function $f(\cdot, \varphi, E, \varepsilon)$ is a consistent solution of (1.3).
- The function $(\varphi, E) \mapsto f(x, \varphi, E, \varepsilon)$ is analytic on $S_{\tilde{Y}} \times \mathcal{V}$ for any $x \in [-X, X]$ and any $\varepsilon \in]0, \varepsilon_0[$.

Then,

1. There exists κ a continuous branch on $\{\operatorname{Re} \varphi < A_g\}$ such that $\operatorname{Im} \kappa > 0$. Moreover, for any $C < B < A_g$, if the function f satisfies the asymptotic behavior

$$f(x, \varphi, E, \varepsilon) = e^{-\frac{i}{\varepsilon} \int^\varphi \kappa(u) du} (\psi_-(x, \varphi, E) + r_C(x, \varphi, E, \varepsilon)) \quad (7.1)$$

with $\lim_{\varepsilon \rightarrow 0} \sup_{[-X, X] \times R_{(-\infty, C]} \times \mathcal{V}} \max\{|r_C(x, \varphi, E, \varepsilon)|, |\partial_x r_C(x, \varphi, E, \varepsilon)|\} = 0$,

then, this behavior stays valid until B . Precisely:

$$f(x, \varphi, E, \varepsilon) = e^{-\frac{i}{\varepsilon} \int^\varphi \kappa(u) du} (\psi_-(x, \varphi, E) + r_B(x, \varphi, E, \varepsilon)) \quad (7.2)$$

with $\lim_{\varepsilon \rightarrow 0} \sup_{[-X, X] \times R_{(-\infty, B]} \times \mathcal{V}} \max\{|r_B(x, \varphi, E, \varepsilon)|, |\partial_x r_B(x, \varphi, E, \varepsilon)|\} = 0$.

2. There exists κ a continuous branch on $\{\operatorname{Re} \varphi > A_d\}$ such that $\operatorname{Im} \kappa > 0$. Moreover, for any $C > B > A_d$, if f satisfies the asymptotic behavior

$$f(x, \varphi, E, \varepsilon) = e^{\frac{i}{\varepsilon} \int^\varphi \kappa(u) du} (\psi_+(x, \varphi, E) + r_C(x, \varphi, E, \varepsilon)) \quad (7.3)$$

with $\lim_{\varepsilon \rightarrow 0} \sup_{[-X, X] \times R_{[C, +\infty)} \times \mathcal{V}} \max\{|r_C(x, \varphi, E, \varepsilon)|, |\partial_x r_C(x, \varphi, E, \varepsilon)|\} = 0$,

then this behavior stays valid until B . Precisely:

$$f(x, \varphi, E, \varepsilon) = e^{\frac{i}{\varepsilon} \int^\varphi \kappa(u) du} (\psi_+(x, \varphi, E) + r_B(x, \varphi, E, \varepsilon)) \quad (7.4)$$

with $\lim_{\varepsilon \rightarrow 0} \sup_{[-X, X] \times R_{[B, +\infty)} \times \mathcal{V}} \max\{|r_B(x, \varphi, E, \varepsilon)|, |\partial_x r_B(x, \varphi, E, \varepsilon)|\} = 0$.

Theorem 4 and Proposition 6 clearly imply Theorem 2.

7.1.2 Some remarks

We shall prove Theorem 4 as W satisfies the weaker assumptions:

(H1) W is an analytic real function in S_Y .

(H2) $\exists C > 0$, $\exists s > 1$ such that $\forall z \in S_Y$, $|W'(z)| \leq \frac{C}{1+|z|^s}$

(H3) $\exists f \in L^1(\mathbb{R})$ such that $\forall x \in \mathbb{R}$ $\sup_{y \in [-Y, Y]} |W(x + iy)| \leq f(x)$

The following lemma relates $(H_{W,r})$ and (H1), (H2) and (H3):

Lemma 7.1. *Let W satisfy $(H_{W,r})$ on S_Y . Fix $\tilde{Y} \in]0, Y[$. Then W satisfies (H1), (H2) and (H3) on $S_{\tilde{Y}}$.*

Proof Assume that W satisfy $(H_{W,r})$ on S_Y . We prove that W satisfies (H2) on $S_{\tilde{Y}}$ by using the following lemma:

Lemma 7.2. *Let f be an analytic function on S_Y such that $|f(z)| \leq \frac{C}{1+|z|^s}$, $C > 0$. Fix $\eta > 0$. Then,*

$$\forall p \in \mathbb{N}^* \quad \exists C_p > 0 / \quad \forall z \in S_{Y-\eta} \quad |f^{(p)}(z)| \leq \frac{C_p}{1+|z|^s}.$$

Proof

This result is a consequence of the Cauchy formula. We do not give the details. \diamond

- Clearly, W satisfies (H1) on S_Y .
- W satisfies (H3) with $f(x) = \frac{C}{1+|x|^s}$.

This completes the proof of Lemma 7.1. \diamond

7.1.3

Let us briefly outline the ideas of the proof. We shall concentrate on $B_g = \{\varphi \in S_Y; \operatorname{Re}(\varphi) < A_g\}$. There are three steps.

First we cover B_g with an union of overlapping local compact canonical domains K_m .

In each canonical domain K_m , we can construct a consistent local basis thanks to Theorem 3. To compute the connection between the consistent bases of K_m and K_{m+n} , it suffices to do the product of the n transfer matrices between the canonical bases of two successive domains. The accuracy of the rest cannot be better than the sum of the accuracies obtained on each domain. Theorem 3 gives an estimate in $o(1)$; this accuracy is insufficient when n goes to infinity.

A refinement of the calculation of asymptotics in Theorem 3 is therefore necessary. We prove it by using the integrability of W .

7.1.4 Branch points

The following result specifies the location of the branch points of κ . We recall that $\Upsilon(E)$ is defined in (2.5).

Lemma 7.3. *Let \mathcal{V} be a complex neighborhood of the interval J . Assume that W satisfies*

$$\lim_{x \rightarrow +\infty} \sup_{y \in [-Y, Y]} |W(x + iy)| = 0,$$

then:

$$\exists A > 0 \text{ such that } \forall E \in \mathcal{V}, \quad \varphi \in \Upsilon(E) \cap S_Y \Rightarrow |\operatorname{Re}(\varphi)| < A.$$

Proof Since $\overline{\mathcal{V}} \cap \partial\sigma(H_0) = \emptyset$, there exists $\alpha > 0$ such that:

$$\forall E \in \mathcal{V}, \quad \forall p \in \mathbb{N}^*, \quad |E - E_p| \geq \alpha.$$

If $\varphi_p(E)$ satisfies $E - W(\varphi_p(E)) = E_p$, we get:

$$\forall E \in \mathcal{V}, \quad \forall p \in \mathbb{N}^*, \quad |W(\varphi_p(E))| \geq \alpha.$$

Finally, $\{u \in S_Y ; |W(u)| \geq \alpha\}$ is a subset of a compact of S_Y . This completes the proof of Lemma 7.3. \diamond

7.1.5 Uniform asymptotics on a δ -chain

First, we introduce a new definition. We remind that the width of a complex subset is defined in (5.9).

Definition 7.1. *δ -chain of strictly canonical domains*

Fix $\tilde{Y} \in]0, Y[$. Fix E . Let D be a simply connected domain of $S_{\tilde{Y}}$ containing no branch points of the complex momentum. We fix on D a continuous branch κ of the complex momentum. Let $\{\tau_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers and K be a compact of $S_{\tilde{Y}}$.

$\{K + \tau_n\}_{n \in \mathbb{N}}$ is called a δ -chain for E , κ and D if it satisfies the following properties:

1. $\bigcup_{n=0}^{\infty} (K + \tau_n) = D$.
2. $\exists \tau > 0$ such that $\forall n \in \mathbb{N} \quad l((K + \tau_n) \cap (K + \tau_{n+1}), \tilde{Y}) > \tau$.
3. The domain K is an union of curves γ such that, for any n , $\gamma + \tau_n$ is a δ -strictly canonical curve for κ .

K is called the fundamental domain of the δ -chain. Now, we have the intermediate result:

Proposition 7. Assume that V satisfies (H_V) and that W satisfies (H_1) , (H_2) and (H_3) . Fix $\tilde{Y} \in]0, Y[$. Let \mathcal{V} a complex neighborhood of J and $D \subset S_{\tilde{Y}}$ a domain with the following properties:

- $\inf_{p \in \mathbb{N}^*, E \in \mathcal{V}} \text{dist}\{D, \varphi_p(E)\} \geq C$,
- there exists $\{\tau_n\}_{n \in \mathbb{N}}$ such that, for any $E \in \mathcal{V}$, $\{K + \tau_n\}_{n \in \mathbb{N}}$ is a δ -chain for E and D .

Fix $\varphi_0 \in D$.

Then, there exists $\varepsilon_0 > 0$ such that, for any $n \in \mathbb{N}$, there exist two functions $(x, \varphi, E, \varepsilon) \mapsto \psi_{\pm}^n(x, \varphi, E, \varepsilon)$ with the following properties:

- The functions $(x, \varphi, E, \varepsilon) \mapsto \psi_{\pm}^n(x, \varphi, E, \varepsilon)$ are defined on $\mathbb{R} \times (K + \tau_n) \times \mathcal{V} \times]0, \varepsilon_0[$ and form a consistent basis.
- for any fixed $x \in \mathbb{R}$, $\varepsilon \in]0, \varepsilon_0[$, the functions $(\varphi, E) \mapsto \psi_{\pm}^n(x, \varphi, E, \varepsilon)$ are analytic on $(K + \tau_n) \times \mathcal{V}$.
- for $x \in [-X, X]$, $\varphi \in (K + \tau_n)$ and $E \in \mathcal{V}$, the functions ψ_{\pm}^n have the asymptotic behavior:

$$\psi_{\pm}^n(x, \varphi, E, \varepsilon) = e^{\pm \frac{i}{\varepsilon} \int_{\varphi_0}^{\varphi} \kappa du} \left(\psi_{\pm}(x, \varphi, E) + \frac{1}{1 + |\tau_n|^s} o(1) \right). \quad (7.5)$$

- The asymptotics (7.5) are uniform in x , τ , $\varphi \in K + \tau_n$ et $E \in \mathcal{V}$.
- The asymptotics can be differentiated once in x .

The proof of Proposition 7 mimics this of Theorem 1.1 in [6]. We omit the details and we refer to [6], section 4 for an analogous statement.

7.2 Construction of a δ -chain of strictly canonical domains

In this section, we shall construct a δ -chain under assumptions (H_1) , (H_2) and (H_3) .

Proposition 8. Fix $\tilde{Y} \in]0, Y[$. Assume that V satisfies (H_V) , that W satisfies (H_1) , (H_2) and (H_3) and that J satisfies (H_J^0) . Then, there exist a complex neighborhood \mathcal{V} of J , two real numbers $(A_g, A_d) \in \mathbb{R}^2$, a domain $K \subset S_{\tilde{Y}}$ and two real sequences $\{\tau_n^1\}_{n \in \mathbb{N}}$, $\{\tau_n^2\}_{n \in \mathbb{N}}$ such that:

- for any $E \in \mathcal{V}$, there exists a continuous branch κ on $\{\varphi \in S_{\tilde{Y}} ; \text{Re } \varphi \in (-\infty, A_g]\}$ (resp. on $\{\varphi \in S_{\tilde{Y}} ; \text{Re } \varphi \in [A_d, \infty)\}$),
- for any $E \in \mathcal{V}$, $\{K + \tau_n^1\}_{n \in \mathbb{N}}$ (resp. $\{K + \tau_n^2\}_{n \in \mathbb{N}}$) is a δ -chain for κ , E and $\{\text{Re } \varphi \in (-\infty, A_g]\}$ (resp. $\{\text{Re } \varphi \in [A_d, +\infty)\}$).

The rest of the section 7.1.5 is devoted to the proof of Proposition 8. This proof is based on elementary geometrical arguments. We prove the construction for $\text{Re } \varphi \in (-\infty, A_g]$.

7.2.1 Construction of δ -strictly canonical straight-lines

We have defined the canonical lines in section 5.1.3 and described them in terms of the vector $t(\varphi)$. We set $\alpha = \frac{1}{2} \inf_{E \in J} \text{Im } k(E)$ and $m = 2 \sup_{E \in J} |\text{Re } k(E)|$.

Since the mapping $(E, \varphi) \mapsto E - W(\varphi)$ is continuous and since $W(\varphi)$ goes to zero when $\text{Re } \varphi$ goes to infinity, there exist a complex neighborhood \mathcal{V} of J and a real number A_g such that:

$$\forall E \in \mathcal{V}, \quad \forall \varphi \in (-\infty, A_g], \quad \text{Re } k(E - W(\varphi)) \in [-m, m], \quad \text{Im } k(E - W(\varphi)) > \alpha$$

We set $B_g = (-\infty, A_g] + i[-\tilde{Y}, \tilde{Y}]$. The canonical curves for $\text{Re } \varphi$ in the neighborhood of $-\infty$ are described by:

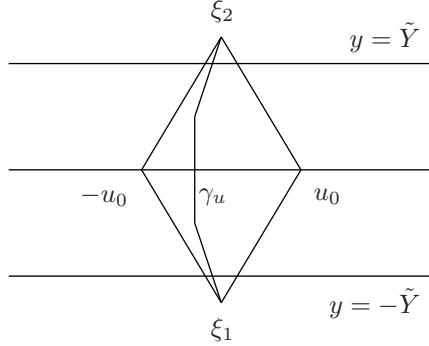


Figure 10: The fundamental domain K

Lemma 7.4. *There exists $\theta_0 \in]0, \pi/2[$ such that, if γ is a smooth curve in B_g satisfying:*

$$\forall \varphi \in \gamma, \quad \arg[t(\varphi)] \in]\theta_0, \pi/2 - \theta_0[. \quad (7.6)$$

then, γ is a canonical line for κ .

Proof For $\arg(u) = \theta$ and $\cot \theta \in]-\frac{m-\delta}{\alpha}, \frac{\pi+m-\delta}{\alpha}[$, we have:

$$\operatorname{Im}(\overline{(\kappa - \delta)u}) > 0 \quad \text{et} \quad \operatorname{Im}(\overline{(\pi - \kappa + \delta)u}) > 0$$

Consequently, $\cot \theta_0 = \frac{m-\delta}{\alpha}$ implies that (7.6) is satisfied. \diamond

7.2.2 The fundamental domain K

Let $\xi_1 = -i\tilde{Y}$ and $\xi_2 = i\tilde{Y}$. We denote by K the lozenge bounded by the straight lines containing ξ_1 and ξ_2 whose guiding vectors have the affixes $e^{i\theta_0}$ and $e^{i(\pi-\theta_0)}$.

We set $[-u_0, u_0] = K \cap \{y = 0\}$. K is shown in figure 10. Fix x such that $K + x \subset B_g$; we shall show that $K + x$ is a δ -strictly canonical domain. According to Lemma 7.4, it suffices to write K as an union of smooth curves satisfying (7.6).

For any $u \in K$, we consider a vertical segment $[\bar{\xi}, \xi]$ containing u and included in K (see figure 10). The broken line $[\xi_1, \bar{\xi}] \cup [\bar{\xi}, \xi] \cup [\xi, \xi_2]$ satisfies (7.6). The relation (7.6) is stable under small C^1 -perturbation; we slightly deform the line $[\xi_1, \bar{\xi}] \cup [\bar{\xi}, \xi] \cup [\xi, \xi_2]$ to get a smooth curve which satisfies (7.6).

Consequently, K satisfies the following properties:

- $K \cap S_{\tilde{Y}}$ contains a rectangle of width $4\eta > 0$.
- $l((K - n\eta) \cap (K - (n+1)\eta), \tilde{Y}) > \eta$.
- K is the union of curves γ such that $\gamma - n\eta$ is δ -strictly canonical for any sufficiently large n .

7.2.3 Conclusion

To finish the proof, it suffices to adapt the proof of Lemma 5.6 in section 5.9 of [6], by using Proposition 8 and Proposition 7. The convergence of the series of general term $\frac{1}{1+|\tau_n|^s}$ replaces the compactness. We do not give the details.

8 Transmission coefficient. Equation for eigenvalues

In Theorem 2, we have constructed two functions h_-^g and h_+^d . We have defined the transmission coefficient $d(E, \varphi, \varepsilon)$. We choose $m_g = -0 + i0$ and $m_d = 0 + i0$.

In Proposition 5, we have introduced a consistent basis (f_i, f_i^*) near the cross. To compute $d(E, \varphi, \varepsilon)$, we shall project the functions h_-^g and h_+^d onto the basis (f_i, f_i^*) .

8.1 Preliminaries

8.1.1 Introduction. Notations

Fix $\tilde{Y} < Y$ and $E_0 \in J$. We have described in section 5 the complex momentum κ and the related geometric objects. We recall that we consider the case (5.15). We use the notations introduced in section 5. The branch points are called φ_r^\pm and $\varphi_i, \overline{\varphi_i}$. We have described the Stokes lines in section 5.10.

We have described in sections 5.9.2 and 5.9.1 the different branches κ_i, κ_g and κ_d . The branch κ_g , resp. κ_d , is defined and continuous on the domain $\{\varphi \in S_Y; \operatorname{Re} \varphi < \varphi_r^-\}$, resp. $\{\operatorname{Re} \varphi > \varphi_r^+\}$. The branch κ_i is defined and continuous on a neighborhood of the cross. The domain $(E - W)(\{\varphi \in S_Y; \operatorname{Re} \varphi < \varphi_r^-\})$ is a simply connected domain which intersects with real axis in only one gap. Thus, we can fix a determination k_g of the quasi-momentum such that:

$$k_g(E - W(\varphi)) = \kappa_g(\varphi).$$

Similarly, we fix the branches k_i and k_d of the quasi-momentum such that :

$$k_i(E - W(\varphi)) = \kappa_i(\varphi), \quad k_d(E - W(\varphi)) = \kappa_d(\varphi).$$

Finally, we set:

$$q_i(\varphi) = \sqrt{k'_i(E - W(\varphi))}, \quad q_g(\varphi) = \sqrt{k'_g(E - W(\varphi))} \quad q_d(\varphi) = \sqrt{k'_d(E - W(\varphi))}.$$

Let $\varphi_g \in \mathbb{R}$ such that $\varphi_g < \varphi_r^-$ and such that the interval $[\varphi_g, \varphi_r^-]$ does not contain any pole of ω_\pm . We define the path γ_g in the complex plane by:

$$\gamma_g = [-0 + i0, \varphi_g + i0] \cup [\varphi_g - i0, -0 - i0].$$

Similarly, fix $\varphi_d \in \mathbb{R}$ such that $\varphi_d > \varphi_r^+$ and such that the interval $[\varphi_r^+, \varphi_d]$ does not contain any pole of ω_\pm . We define the path γ_d in the complex plane by:

$$\gamma_d = [0 + i0, \varphi_d + i0] \cup [\varphi_d - i0, 0 - i0].$$

In the following section, we explain the choice of the determinations q_i, q_g and q_d .

8.1.2 The determination q

We recall that there exists a real number $\sigma_i \in \{-1, 1\}$ such that:

$$\frac{q_i^*}{q_i} = \sigma_i. \tag{8.1}$$

We refer to section 5.11.6.

The Wronskian satisfies $w(f_i, (f_i)^*) = \sigma_i(w_0 k'_i)(E - W(0))$.

The number σ_i depends on the sign of k' along the band B :

- If the band B can be written $[E_{4p+1}, E_{4p+2}]$, then $k' > 0$ on B and $\sigma_i = 1$.
- If the band B can be written $[E_{4p+3}, E_{4p+4}]$, then $k' < 0$ on B and $\sigma_i = -1$.

We fix the branch q_g such that $q_g = q_i$ in S_- and such that q_g is analytically continued in $\{\varphi \in S_Y; \operatorname{Re} \varphi < \varphi_r^-\}$. According to relation (8.1), the branch q_g satisfies:

$$q_g^* = i\sigma_i q_g \quad (8.2)$$

Similarly, we fix q_d such that $q_d = q_i$ in $\overline{S_+}$ and such that q_d is analytically continued in $\{\varphi \in S_Y; \operatorname{Re} \varphi > \varphi_r^+\}$. The branch q_d satisfies:

$$q_d^* = i\sigma_i q_d \quad (8.3)$$

According to equations (8.2) and (8.3), we have also:

$$\sigma_g = \sigma_i \quad ; \quad \sigma_d = \sigma_i.$$

We denote by $\widetilde{\Psi}_\pm^g(x, \mathcal{E})$, $\widetilde{\Psi}_\pm^i(x, \mathcal{E})$ and $\widetilde{\Psi}_\pm^d(x, \mathcal{E})$ the Bloch solutions described in section 4.1. We set:

$$\Psi_\pm^i(x, \varphi, E) = \widetilde{\Psi}_\pm^i(x, E - W(\varphi)) ; \quad \Psi_\pm^g(x, \varphi, E) = \widetilde{\Psi}_\pm^g(x, E - W(\varphi)) ; \quad \Psi_\pm^d(x, \varphi, E) = \widetilde{\Psi}_\pm^d(x, E - W(\varphi)).$$

We define the functions ω_\pm^g , ω_\pm^i and ω_\pm^d associated by (5.7) to the branches k_g , k_i and k_d .

8.1.3 Ideas of the method

The computation is similar to this done in [8, 6, 9]. It is based on some elementary principles that we outline now.

1. Periodicity.

The consistency condition (1.5) implies that the Wronskians are ε -periodic in φ . To get a total control of the Wronskians in a horizontal strip, we only need to control them in some vertical sub-strip of width ε .

2. Analyticity.

Since the functions $(\varphi, E) \mapsto f_-^g(x, \varphi, E, \varepsilon)$, $(\varphi, E) \mapsto f_+^d(x, \varphi, E, \varepsilon)$, $(\varphi, E) \mapsto f_\pm^i(x, \varphi, E, \varepsilon)$ are analytic on $S_{\tilde{Y}} \times \mathcal{U}$, their Wronskians are analytic in $(\varphi, E) \in S_{\tilde{Y}} \times \mathcal{U}$. This allows us to expand them into exponentially converging series.

Let $w(\varphi, E, \varepsilon)$ be an analytic function in (φ, E) which is ε -periodic in φ . We set:

$$w(\varphi, E, \varepsilon) = \sum_{k \in \mathbb{Z}} w_k(E, \varepsilon) e^{\frac{2i\pi\varphi}{\varepsilon}}$$

The Cauchy formula gives an estimate of the Fourier coefficients:

$$w_k(E, \varepsilon) = \frac{1}{\varepsilon} \int_{\varphi_0}^{\varphi_0 + \varepsilon} w(\varphi, E, \varepsilon) e^{-\frac{2ik\pi\varphi}{\varepsilon}} d\varphi, \quad \forall k \in \mathbb{N}, \quad \forall \varphi_0 \in S_{\tilde{Y}}. \quad (8.4)$$

By moving $\operatorname{Im} \varphi_0$ in $[-\tilde{Y}, \tilde{Y}]$, we get a control of positive and negative coefficients.

8.2 Asymptotic expansion of $d(\varphi, E, \varepsilon)$

In this section, we shall establish the following result.

Proposition 9. *For any E_0 in J , there exist a complex neighborhood \mathcal{U}_0 of E_0 and two functions $(\varphi, E, \varepsilon) \mapsto b_g^-(\varphi, E, \varepsilon)$ and $(\varphi, E, \varepsilon) \mapsto b_d^+(\varphi, E, \varepsilon)$ such that:*

- The coefficient d defined in (6.26) can be written:

$$d(\varphi, E, \varepsilon) = i\sigma_i w(f_i, (f_i)^*) [b_g^-(b_d^+)^* - (b_g^-)^* b_d^+]. \quad (8.5)$$

- The functions $(\varphi, E) \mapsto b_g^-(\varphi, E, \varepsilon)$ and $(\varphi, E) \mapsto b_d^+(\varphi, E, \varepsilon)$ are analytic on $S_Y \times \mathcal{U}_0$.

- The functions $\varphi \mapsto b_g^-(\varphi, E, \varepsilon)$ and $\varphi \mapsto b_d^+(\varphi, E, \varepsilon)$ are ε -periodic and admit the following Fourier asymptotic expansion, when $\varepsilon \rightarrow 0$:

$$b_g^-(\varphi, E, \varepsilon) = \sum_{k \in \mathbb{Z}} (b_g^-)_k(E, \varepsilon) e^{\frac{2ik\pi\varphi}{\varepsilon}}, \quad (8.6)$$

with

$$(b_g^-)_0(E, \varepsilon) = \sigma_i e^{-\frac{i}{\varepsilon} \int_0^{\varphi_r^-} \kappa_i} e^{\frac{1}{2} \int_0^{\varphi_r^-} (\omega_+^i - \omega_-^i)} [1 + o(1)], \quad (8.7)$$

and

$$\forall k \neq 0, \quad |(b_g^-)_k(E, \varepsilon)| < C e^{-\alpha/\varepsilon} e^{\frac{-2|k|\pi Y_0}{\varepsilon}}, \quad (8.8)$$

$$b_d^+(\varphi, E, \varepsilon) = \sum_{k \in \mathbb{Z}} (b_d^+)_k(E, \varepsilon) e^{\frac{2ik\pi\varphi}{\varepsilon}}, \quad (8.9)$$

with

$$(b_d^+)_0(E, \varepsilon) = i \sigma_i e^{\frac{i}{\varepsilon} \int_0^{\varphi_r^+} \kappa_i} e^{\frac{1}{2} \int_0^{\varphi_r^+} (\omega_+^i - \omega_-^i)} [1 + o(1)], \quad (8.10)$$

$$(b_d^+)_1(E, \varepsilon) = -i \sigma_i e^{\frac{i}{\varepsilon} \int_{\varphi_r^+}^0 \kappa_i} e^{\frac{2i}{\varepsilon} \int_0^{\varphi_r^+} (\kappa_i - \pi)} e^{\frac{1}{2} \int_{\varphi_r^+}^0 \omega_+^i - \omega_-^i} e^{\int_0^{\varphi_r^+} (\omega_+^i - \omega_-^i)} [1 + o(1)], \quad (8.11)$$

et

$$\forall k > 1, \quad |(b_d^+)_k(E, \varepsilon)| < C |(b_d^+)_1(E, \varepsilon)| e^{-\alpha/\varepsilon} e^{\frac{-2|k-1|\pi Y_0}{\varepsilon}}, \quad (8.12)$$

$$\forall k < 0, \quad |(b_d^+)_k(E, \varepsilon)| < C e^{-\alpha/\varepsilon} e^{\frac{-2|k|\pi Y_0}{\varepsilon}}, \quad (8.13)$$

The rest of the section is devoted to the proof of Proposition 9.

Fix $E_0 \in J$. According to the choice of κ_g and κ_d (sections 5.9.2 and 5.9.1), there exist two analytic functions $\alpha_g(E)$ and $\alpha_d(E)$ such that:

$$(\alpha_g h_-^g)^* = i \sigma_g \alpha_g h_-^g,$$

$$(\alpha_d h_+^d)^* = i \sigma_d \alpha_d h_+^d.$$

Now, we use the function f_i constructed in Proposition 5. There exists a neighborhood \mathcal{U}_0 of E_0 such that we can write:

$$\alpha_g h_-^g = -i \sigma_g (b_g^-)^* f_i + b_g^-(f_i)^*,$$

$$\alpha_d h_+^d = -i \sigma_d (b_d^+)^* f_i + b_d^+(f_i)^*.$$

The coefficients α_g and α_d are defined in equations (6.22) and (6.25). We compute:

$$\int_{\gamma_g} \omega_-^g = \int_0^{\varphi_r^-} (\omega_+^i - \omega_-^i),$$

$$\int_{\gamma_d} \omega_+^d = \int_0^{\varphi_r^+} (\omega_+^i - \omega_-^i).$$

This leads to:

$$\alpha_g(E) = e^{-\frac{i}{\varepsilon} \int_0^{\varphi_r^-} \kappa_i} e^{\frac{1}{2} \int_0^{\varphi_r^-} (\omega_+^i - \omega_-^i)},$$

$$\alpha_d(E) = e^{\frac{i}{\varepsilon} \int_0^{\varphi_r^+} \kappa_i} e^{\frac{1}{2} \int_0^{\varphi_r^+} (\omega_+^i - \omega_-^i)}.$$

The coefficients b_g^- and b_d^+ satisfy:

$$b_g^- = \alpha_g a_g^-, \quad b_d^+ = \alpha_d a_d^+,$$

where the coefficients a_g^- and a_d^+ are given by:

$$a_g^- = \frac{w(f_i, h_-^g)}{w(f_i, f_i^*)}, \quad (8.14)$$

and:

$$a_d^+ = \frac{w(f_i, h_+^d)}{w(f_i, f_i^*)}. \quad (8.15)$$

We compute, for $E \in \mathcal{U}_0$:

$$\begin{aligned} d(\varphi, E, \varepsilon) &= w(\alpha_g h_-^g, \alpha_d h_+^d) \\ &= [b_g^-(b_d^+)^* - b_d^+(b_g^-)^*] i\sigma_i w(f_i, (f_i)^*). \end{aligned}$$

8.2.1 Continuation diagram of f_i

First, we describe the asymptotic behavior of the function f_i in some domains of the complex plane.

Lemma 8.1. *We suppose that the assumptions of Proposition 5 are satisfied. Fix $\tilde{Y} < Y$. Fix $\varphi_g < \varphi_r^-$ and $\varphi_d > \varphi_r^+$. There exists $y_0 \in]0, \text{Im } \varphi_i[$ such that the function f_i has the following asymptotic behavior:*

- For $\varphi \in \{\varphi \in S_{\tilde{Y}}; \text{Re } \varphi \in [\varphi_g, \varphi_r^-]\}$, f_i has the standard asymptotic behavior:

$$f_i = q_g e^{\frac{i}{\varepsilon} \int_0^{\varphi_r^-} \kappa_i} e^{\frac{i}{\varepsilon} \int_{\varphi_r^-}^{\varphi} \kappa_g} e^{\int_0^{\varphi_r^-} \omega_+^i} e^{\int_{\varphi_r^-}^{\varphi} \omega_+^g} (\Psi_+^g + o(1)).$$

- For $\varphi \in \{\varphi \in S_{\tilde{Y}}; \text{Re } \varphi \in [\varphi_r^+, \varphi_d]; \text{Im } \varphi > -y_0\}$, f_i has the standard asymptotic behavior:

$$f_i = i q_d e^{\frac{i}{\varepsilon} \int_0^{\varphi_r^+} \kappa_i} e^{-\frac{i}{\varepsilon} \int_{\varphi_r^+}^{\varphi} \kappa_d} e^{\int_0^{\varphi_r^+} \omega_+^i} e^{\int_{\varphi_r^+}^{\varphi} \omega_+^d} (\Psi_-^d + o(1)).$$

- For $\varphi \in \{\varphi \in S_{\tilde{Y}}; \text{Re } \varphi \in [\varphi_r^+, \varphi_d]; \text{Im } \varphi < -y_0\}$, f_i has the standard asymptotic behavior:

$$f_i = -i q_d e^{\frac{i}{\varepsilon} \int_0^{\varphi_i^-} \kappa_i} e^{\frac{i}{\varepsilon} \int_{\varphi_i^-}^{(2\pi - \kappa_d)} \omega_+^i} e^{\int_0^{\varphi_i^-} \omega_+^i} e^{\int_{\varphi_i^-}^{\varphi} \omega_+^d} (\Psi_-^d + o(1)).$$

Proof This lemma is similar to the continuation diagram presented in section 6 of [9]. Thus, we give only the main ideas of the study and refer to this paper for the details. The continuation diagram is represented in figure 11. In this figure, the straight arrows indicate the use of continuation lemma (Lemma 5.6), the circular arrows the use of the Stokes lemma (Lemma 5.7) and the hatched zones the use of the Adjacent Canonical Domain Principle (Lemma 4). To complete the proof, it remains to explain the connections between the different objects of the WKB method.

- According to the definitions given in section 5.9.2, the branches κ_i and κ_g are equal in S_- and, for all $\varphi \in S_-$, we have:

$$\kappa_i(\varphi + 0) = \kappa_g(\varphi - 0), \quad \Psi_{\pm}^i(\varphi + 0) = \Psi_{\pm}^g(\varphi - 0), \quad \omega_{\pm}^i(\varphi + 0) = \omega_{\pm}^g(\varphi - 0). \quad (8.16)$$

Besides, it remains to link q_i and q_g . Section 8.1.2 implies:

$$\forall \varphi \in S_-, \quad q_i(\varphi) = q_g(\varphi).$$

- Similarly, we have, for all $\varphi \in \overline{S_+}$:

$$\kappa_i(\varphi - 0) = -\kappa_d(\varphi + 0), \quad \Psi_{\pm}^i(x, \varphi - 0) = \Psi_{\mp}^d(x, \varphi + 0), \quad (8.17)$$

$$\omega_{\pm}^i(\varphi - 0) = \omega_{\mp}^d(\varphi + 0), \quad q_d(\varphi + 0) = -i q_i(\varphi - 0)$$

- We study finally the link between κ_i and κ_d along the Stokes line \bar{c} beginning at $\bar{\varphi}_i$. We consider the quasi-momenta k_d and k_i associated to κ_d and κ_i . Equation (4.12) for k_d and k_i , on either side of $[E_2, E_3]$, implies that κ_d and κ_i satisfy the following relations, for $\varphi \in c$,

$$\kappa_d(\varphi + 0) = 2\pi - \kappa_i(\varphi - 0), \quad \Psi_{\pm}^d(x, \varphi + 0) = \Psi_{\mp}^i(x, \varphi - 0) \quad (8.18)$$

$$\omega_{\pm}^d(\varphi + 0) = \omega_{\mp}^i(\varphi - 0) \quad q_d(\varphi + 0) = i q_i(\varphi - 0)$$

◇

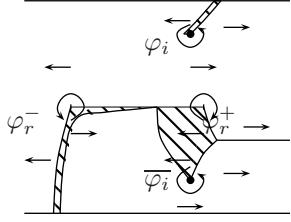


Figure 11: Continuation Diagram

8.2.2 Computation of b_g^- and b_d^+

Now, we compute the coefficients b_g^- and b_d^+ given by (8.14) and (8.15).

According to Theorem 4, we know that the asymptotic behavior of the function h_g^g remains valid in the domain $\{\varphi \in S_Y; \operatorname{Re}(\varphi) \in [\varphi_g, \varphi_r^-]\}$. Lemma 8.1 gives the asymptotic behavior of f_i in this domain and we get :

$$\forall \varphi \in S_Y, \quad a_g^-(\varphi, E, \varepsilon) = \sigma_i [1 + o(1)]. \quad (8.19)$$

Fix $Y_0 \in]0, Y[$. In the strip S_{Y_0} , we write:

$$a_g^-(\varphi, E, \varepsilon) = \sum_{n \in \mathbb{Z}} \alpha_n e^{\frac{2i\pi n \varphi}{\varepsilon}} \quad (8.20)$$

The coefficients α_n satisfy:

$$\alpha_n = \frac{1}{\varepsilon} \int_{\varphi_0}^{\varphi_0 + \varepsilon} a_g^-(\varphi, E, \varepsilon) e^{-2i\pi n \frac{\varphi}{\varepsilon}} d\varphi, \quad \forall n \in \mathbb{N}, \quad \forall \varphi_0 \in \{-Y_0 \leq \operatorname{Im} \varphi \leq Y_0\}. \quad (8.21)$$

Fix $n > 0$. We estimate $|\alpha_n|$. We use formula (8.21) for $\operatorname{Im} \varphi_0 = -(Y - \delta)$, and we get:

$$|\alpha_n| \leq C e^{-2\pi n(Y - \delta)/\varepsilon} e^{\frac{C\delta}{\varepsilon}}.$$

We treat similarly the case $n < 0$ with $\operatorname{Im} \varphi_0 = (Y - \delta)$ and we obtain:

$$|\alpha_n| \leq C e^{2\pi n(Y - \delta)/\varepsilon} e^{\frac{C\delta}{\varepsilon}}.$$

Besides, we have:

$$\alpha_0 = \sigma_i [1 + o(1)].$$

We fix $\delta < \frac{2\pi(Y - Y_0)}{C + 2\pi}$. For a constant C such that $\alpha < 2\pi(Y - Y_0) - \delta(C + 2\pi)$, we obtain the estimates (8.7) and (8.8).

The arguments for the coefficients a_d^+ and b_d^+ are similar.

8.3 Proof of Lemma 2.2

Now, we want to express the coefficient d in a more understandable form. We begin with proving Lemma 2.2. We recall that we denote by φ_r^\pm and φ_i , $\overline{\varphi_i}$ the branch points of the complex momentum, and by E_r and E_i the related ends of $\sigma(H_0)$. We shall prove the lemma in the case (5.15). Let κ_i be the branch described in section 5.9.1. κ_i satisfies (5.21).

We shall prove Lemma 2.2 for the branch $\tilde{\kappa}_i = \kappa_i$.

- First, we express Φ , S and Φ_d as integrals of the complex momentum along complex paths. Let γ be an oriented curve, we call γ^\dagger the curve oriented in the opposite direction. Fix $\varphi_d \in \mathbb{R}$ and $\varphi_g \in \mathbb{R}$ such that:

$$\varphi_d > \varphi_r^+ ; \varphi_g < \varphi_r^-.$$

We define the complex paths γ_Φ , γ_S and $\gamma_{g,d}$:

$$\gamma_\Phi = [\varphi_r^- + i0, \varphi_r^+ + i0] \cup [\varphi_r^+ - i0, \varphi_r^- - i0],$$

$$\gamma_S = (\sigma + 0) \cup (\sigma^\dagger - 0),$$

$$\gamma_{g,d} = [\varphi_g + i0, 0 + i0] \cup (\sigma_+ - 0) \cup (\sigma_+^\dagger + 0) \cup [0 + i0, \varphi_d + i0].$$

These paths are represented in figure 12. We have the following result:

Lemma 8.2. *The coefficients Φ , Φ_d and S can be written:*

$$\Phi = \frac{1}{2} \oint_{\gamma_\Phi} \kappa(u) du,$$

$$S = \frac{1}{2i} \oint_{\gamma_S} \kappa(u) du,$$

$$\Phi_d = \frac{1}{2} \left(\int_{\gamma_{g,d}} (\kappa(u) - \pi) du + \int_{\overline{\gamma_{g,d}}} (\tilde{\kappa}(u) - \pi) du \right) + \pi(\varphi_g - \varphi_d).$$

where $\kappa = \kappa_i$ in S_- and κ is analytically continued along each path; $\tilde{\kappa} = \kappa_i$ in $\overline{S_-}$ and $\tilde{\kappa}$ is analytically continued along $\overline{\gamma_{g,d}}$.

Proof

– First, let us justify the fact that integrals along γ_Φ and γ_S can be considered along closed curves.

It suffices to show that κ can be analytically continued along γ_Φ and γ_S .

We consider the curve γ_Φ . We have taken the cut of γ_Φ in φ_r^- . We show that κ has the same values on each side of the cut. $\kappa = \kappa_i$ on $[\varphi_r^- + i0, \varphi_r^+ + i0]$, since κ is continuous to the right of φ_r^+ , we obtain that $\kappa = -\kappa_i$ on $[\varphi_r^- - i0, \varphi_r^+ - i0]$. In addition, $\kappa(\varphi_r^- + i0) = 0 = \kappa(\varphi_r^- - i0)$, which proves that the integral can be taken on the closed curve γ_Φ .

The arguments for γ_S are similar.

– We compute:

$$\frac{1}{2} \oint_{\gamma_\Phi} \kappa(u) du = \int_{\varphi_r^-}^{\varphi_r^+} \kappa_i(u) du = \Phi(E).$$

Similarly, for the coefficient $S(E)$,

$$\frac{1}{2i} \oint_{\gamma_S} \kappa(u) du = \frac{1}{i} \left[\int_{\sigma_+} (\pi - \kappa_i(u)) du + \int_{\sigma_-} (\pi - \kappa_i(u)) du \right].$$

– It remains to study Φ_d . We introduce the branch κ_i and we cut $\gamma_{g,d}$ in elementary segments:

$$\begin{aligned} & \int_{\gamma_{g,d}} (\kappa(u) - \pi) du + \int_{\overline{\gamma_{g,d}}} (\tilde{\kappa}(u) - \pi) du \\ &= 2 \int_{\varphi_g^-}^0 (\kappa_i(u) - \pi) du + 2 \int_{\sigma_+} (\kappa_i(u) - \pi) du + \int_{\varphi_d}^0 (\kappa_i(u) - \pi) du - 2 \int_{\sigma_-} (\kappa_i(u) - \pi) du \\ &= 2 \int_{\varphi_r^-}^0 (\kappa_i(u) - \pi) du + 2 \int_{\sigma_+} (\kappa_i(u) - \pi) du + \int_{\varphi_r^+}^0 (\kappa_i(u) - \pi) du - 2 \int_{\sigma_-} (\kappa_i(u) - \pi) du - 2\pi(\varphi_g - \varphi_r^-) + 2\pi(\varphi_d - \varphi_r^+) \\ &= 2\Phi_d(E) + 2\pi(\varphi_d - \varphi_g) \end{aligned}$$

This ends the proof of Lemma 8.2. \diamond

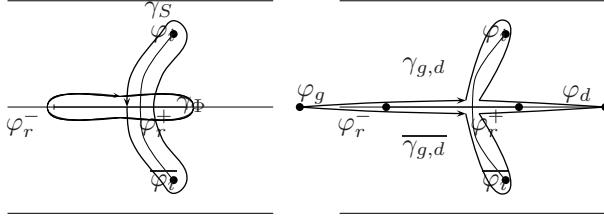


Figure 12: Some complex paths

- We use Lemma 8.2 to prove the analyticity of Φ , S and Φ_d .

First, we consider Φ . We can deform γ_Φ to a closed curve going around $[\varphi_r^-, \varphi_r^+]$ and staying at a nonzero distance from this interval. Besides, κ is analytic in E on the integration contour when E is close enough to J . The analysis of the coefficient S is done in the same way. To prove that Φ_d is analytic, we deform the curves $\gamma_{g,d}$ and $\overline{\gamma_{g,d}}$ to stay at a nonzero distance of the cross.

- Fix $E \in J$. On the interval $[\varphi_r^-, \varphi_r^+]$, the branch κ_i satisfies $\kappa_i \in [0, \pi]$. Thus, the function $\Phi(E)$ is real positive on J .

Now, we give a simplified expression of S :

$$S(E) = -i \left[\int_{\sigma_+} (\pi - \kappa_i(u)) du + \int_{\sigma_-} (\pi - \kappa_i(u)) du \right] = 2\text{Im} \int_{\sigma_+} (\pi - \kappa_i(u)) du \quad (8.22)$$

On σ , the branch κ_i satisfies $\kappa_i \in [0, \pi]$. According to (2.8) and (8.22), we obtain that $0 < S(E) \leq 2\pi\text{Im} \varphi_i(E)$.

Finally, we have $\int_{\sigma_-} (\kappa_i(u) - \pi) du = -\overline{\int_{\sigma_+} (\kappa_i(u) - \pi) du}$. Consequently, the coefficient $\Phi_d(E)$ is real.

- Now, we compute S' and Φ' on J . Let k be the branch of the Bloch momentum continuous through $[E_r, E_i]$, then $\kappa(\varphi) = k(E - W(\varphi))$ and

$$\Phi'(E) = \int_{\varphi_r^-}^{\varphi_r^+} k'(E - W(u)) du + k(E - W(\varphi_r^+)) - k(E - W(\varphi_r^-)) = \int_{\varphi_r^-}^{\varphi_r^+} k'(E - W(u)) du.$$

We recall that k has some branch points of square root type at the ends of spectral bands (see section 4.9); consequently, the integral $\int_{\varphi_r^-}^{\varphi_r^+} k'(E - W(u)) du$ is convergent. In the interval $[E_r, E_i]$, $k'(\mathcal{E}) > 0$ and $(E_i - E_r)\Phi'$ takes positive values on J .

The analysis of S' is similar.

- We complete this section with the following formulas:

$$\Phi_d(E) + iS(E) = \int_{\varphi_r^-}^0 \kappa_i(u) du - 2 \int_{\sigma_-} (\kappa_i - \pi)(u) du + \int_{\varphi_r^+}^0 \kappa_i(u) du \quad (8.23)$$

$$-\Phi_d(E) + iS(E) = -\int_{\varphi_r^-}^0 \kappa_i(u) du - 2 \int_{\sigma_+} (\kappa_i - \pi)(u) du - \int_{\varphi_r^+}^0 \kappa_i(u) du \quad (8.24)$$

When $\kappa(\varphi_r^-) = \pi$, the proof is analogous for the branch $\tilde{\kappa}_i = 2\pi - \kappa_i$.

8.3.1 Further computations

We recall that the functions ω_+^i and ω_-^i are defined in (5.7). We consider the integrals of ω_+^i and ω_-^i along some paths of the complex plane. We have the following relations:

Lemma 8.3. *The integrals of ω_+^i and ω_-^i satisfy:*

$$\forall E \in J, \quad \int_{[\varphi_r^-, \varphi_r^+]} \omega_+^i(u, E) du = 0, \quad \int_{[\varphi_r^-, \varphi_r^+]} \omega_-^i(u, E) du = 0 \quad (8.25)$$

$$\forall E \in J, \quad \int_{\sigma} \omega_+^i(u, E) du = 0, \quad \int_{\sigma} \omega_-^i(u, E) du = 0 \quad (8.26)$$

There exists a real number ρ such that:

$$\forall E \in J, \quad \int_{[\varphi_r^+, 0] \cup \sigma_+} (\omega_+^i(u, E) - \omega_-^i(u, E)) du - \int_{\sigma_- \cup [0, \varphi_r^-]} (\omega_+^i(u, E) - \omega_-^i(u, E)) du = i\rho \quad (8.27)$$

Proof We consider the case (5.15).

- We first prove (8.25). According to (5.7), we compute:

$$\int_{[\varphi_r^-, \varphi_r^+]} \omega_+^i(u, E) du = - \int_{[\varphi_r^-, \varphi_r^+]} g_+^i(E - W(u)) W'(u) du = \int_{E - W([\varphi_r^-, \varphi_r^+])} g_+^i(e) de = 0$$

Indeed, for $E \in J$, the subset $E - W([\varphi_r^-, \varphi_r^+])$ is a complex path of energies connecting E_r to E_r and containing $(E - W(0)) \in]E_1, E_2[$. We have shown this path in figure 13A. Particularly, $E - W([\varphi_r^-, \varphi_r^+])$ is a closed path and does not surround any pole of the meromorphic function g_+^i . Consequently, the integral is zero. We prove similarly that

$$\int_{[\varphi_r^-, \varphi_r^+]} \omega_-^i(u, E) du = 0.$$

- We consider now (8.26). We write:

$$\int_{\sigma} \omega_+^i(u, E) du = - \int_{E - W(\sigma)} g_+^i(e) de$$

The image of the path σ is shown in figure 13B. We deal with ω_-^i similarly.

- Finally, we compute:

$$\begin{aligned} & \int_{[\varphi_r^+, 0] \cup \sigma_+} (\omega_+^i(u, E) - \omega_-^i(u, E)) du - \int_{\sigma_- \cup [0, \varphi_r^-]} (\omega_+^i(u, E) - \omega_-^i(u, E)) du \\ &= \int_{E - W([\varphi_r^+, 0] \cup \sigma_+)} (g_+^i(e) - g_-^i(e)) de - \int_{E - W(\sigma_- \cup [0, \varphi_r^-])} (g_+^i(e) - g_-^i(e)) de \end{aligned}$$

The images $E - W([\varphi_r^+, 0] \cup \sigma_+)$ and $E - W(\sigma_- \cup [0, \varphi_r^-])$ are two paths of energies connecting E_r to E_i (see figure 13C). By analyticity of $(g_+^i - g_-^i)$ in the domain $\text{Re}(e) \in]E_r, E_i[$, we obtain that:

$$\int_{[\varphi_r^+, 0] \cup \sigma_+} (\omega_+^i(u, E) - \omega_-^i(u, E)) du - \int_{\sigma_- \cup [0, \varphi_r^-]} (\omega_+^i(u, E) - \omega_-^i(u, E)) du = 2 \int_{E_r}^{E_i} (g_+^i - g_-^i)(e) de \quad (8.28)$$

It remains to show that this coefficient is purely imaginary. To do that, we point out that $(g_-^i)^* = g_+^i$, according to (4.18). Equation (8.28) becomes:

$$\begin{aligned} & \int_{[\varphi_r^+, 0] \cup \sigma_+} (\omega_+^i - \omega_-^i)(u, E) du - \int_{\sigma_- \cup [0, \varphi_r^-]} (\omega_+^i - \omega_-^i)(u, E) du \\ &= 2 \left(\int_{E_r}^{E_i} g_+^i(e) de - \int_{E_r}^{E_i} (g_+^i)^*(e) de \right) = 2 \left(\int_{E_r}^{E_i} g_+^i(e) de - \overline{\int_{E_r}^{E_i} g_+^i(e) de} \right) \end{aligned}$$

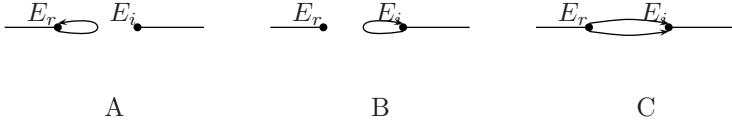


Figure 13: Paths in the complex plane of energy

This ends the proof of Lemma 8.3. \diamond

8.4 Equation for the eigenvalues

The following result gives a characterization of the eigenvalues of $H_{\varphi, \varepsilon}$.

Proposition 10. *We assume that (H_V) , $(H_{W,r})$, $(H_{W,g})$ and (H_J) are satisfied.*

There exist $\varepsilon_0 > 0$, a neighborhood $\mathcal{V} = \overline{\mathcal{V}}$ of J , two functions $(E, \varepsilon) \mapsto \tilde{\Phi}(E, \varepsilon)$ and $(E, \varepsilon) \mapsto \tilde{\Phi}_d(E, \varepsilon)$ defined on $\mathcal{V} \times]0, \varepsilon_0[$ and two functions $(\varphi, E, \varepsilon) \mapsto F(\varphi, E, \varepsilon)$ and $(\varphi, E, \varepsilon) \mapsto R_2(\varphi, E, \varepsilon)$ defined on $\mathbb{R} \times \mathcal{V} \times]0, \varepsilon_0[$ such that:

1. *E is an eigenvalue of $H_{\varphi, \varepsilon}$ if and only if:*

$$F(\varphi, E, \varepsilon) = 0$$

2. *The function F satisfies:*

$$\forall \varphi \in \mathbb{R}, \forall E \in \mathcal{V}, \forall \varepsilon \in]0, \varepsilon_0[, \quad F^*(\varphi, E, \varepsilon) = \overline{F(\overline{\varphi}, \overline{E}, \varepsilon)} = F(\varphi, E, \varepsilon).$$

3. *The function $\varphi \mapsto F(\varphi, E, \varepsilon)$ is ε -periodic and its Fourier expansion is written:*

$$F(\varphi, E, \varepsilon) = \cos\left(\frac{\tilde{\Phi}(E)}{\varepsilon}\right) + e^{-S(E)/\varepsilon} \cos\left(\frac{\tilde{\Phi}_d(E)}{\varepsilon} + \frac{2\pi\varphi}{\varepsilon} + \rho\right) + e^{-S(E)/\varepsilon} R_2(\varphi, E, \varepsilon) \quad (8.29)$$

4. *The functions $\tilde{\Phi}$, $\tilde{\Phi}_d$ satisfy the following properties for any $\varepsilon \in]0, \varepsilon_0[$:*

- $E \mapsto \tilde{\Phi}(E, \varepsilon)$ and $E \mapsto \tilde{\Phi}_d(E, \varepsilon)$ are analytic on \mathcal{V} .
- $\tilde{\Phi}(E, \varepsilon) = \Phi(E) + o(\varepsilon)$ and $\tilde{\Phi}_d(E, \varepsilon) = \Phi_d(E) + o(\varepsilon)$ uniformly for $E \in \mathcal{V}$.

5. *For any $\varepsilon \in]0, \varepsilon_0[$, the function $(\varphi, E) \mapsto R_2(\varphi, E, \varepsilon)$ is analytic on $\mathbb{R} \times \mathcal{V}$. Besides, there exists a constant $\alpha > 0$ such that, for all $\varepsilon \in]0, \varepsilon_0[$, and all E in \mathcal{V} , the function R_2 satisfies the following properties:*

$$\int_0^\varepsilon R_2(u, E, \varepsilon) du = 0, \quad \int_0^\varepsilon R_2(u, E, \varepsilon) e^{\frac{2i\pi u}{\varepsilon}} du = 0, \quad \int_0^\varepsilon R_2(u, E, \varepsilon) e^{\frac{-2i\pi u}{\varepsilon}} du = 0,$$

$$\sup_{\varphi \in \mathbb{R}, E \in \mathcal{V}} |R_2(\varphi, E, \varepsilon)| \leq e^{-\frac{\alpha}{\varepsilon}}$$

The functions Φ , Φ_d , S are defined in Lemma 2.2. ρ is a real number defined in (8.27).

Now, we prove Proposition 10.

- Now, it suffices to compute the Fourier expansion of:

$$b_g^-(b_d^+)^*(\varphi, E, \varepsilon) = \sum_{n \in \mathbb{Z}} \gamma_n(E, \varepsilon) e^{\frac{2in\pi\varphi}{\varepsilon}}.$$

By using the asymptotic expansion of the coefficients b_g^- and b_d^+ given in Lemma 9, we prove that:

$$\begin{aligned}\gamma_0 &= -ie^{-\frac{i}{\varepsilon} \int_{\varphi_r^-}^{\varphi_r^+} \kappa_i} [1 + o(1)]. \\ \gamma_1 &= +ie^{-\frac{i}{\varepsilon} (\int_{\varphi_r^+}^0 \kappa_i + \int_{\varphi_r^-}^0 \kappa_i)} e^{\frac{2i}{\varepsilon} \int_0^{\varphi_r^+} (\kappa_i - \pi)} e^{\int_0^{\varphi_r^+} (\omega_+^i - \omega_-^i)} e^{\int_0^{\varphi_r^-} (\omega_-^i - \omega_+^i)} [1 + o(1)]. \\ &\quad \left| \sum_{n \in \mathbb{Z} \setminus \{0,1\}} \gamma_n e^{\frac{2in\pi\varphi}{\varepsilon}} \right| = O(e^{-\alpha/\varepsilon}) \quad \text{pour } \varphi \in S_{Y_0}.\end{aligned}$$

Actually,

$$\gamma_0 = \alpha_0 \beta_0 + \sum_{n \neq 0} \alpha_n \beta_{-n} = -ie^{-\frac{i}{\varepsilon} \int_{\varphi_r^-}^{\varphi_r^+} \kappa_i} e^{\frac{1}{2} \left[\int_0^{\varphi_r^-} (\omega_+^i - \omega_-^i) + \int_0^{\varphi_r^+} (\omega_-^i - \omega_+^i) \right]} [1 + o(1)].$$

According to (8.25), we simplify:

$$\left[\int_0^{\varphi_r^-} (\omega_+^i - \omega_-^i) + \int_0^{\varphi_r^+} (\omega_-^i - \omega_+^i) \right] = 0.$$

According to (2.7), $\int_{\varphi_r^-}^{\varphi_r^+} \kappa_i = \Phi(E)$. Consequently,

$$\gamma_0 = -ie^{\frac{i\Phi(E)}{\varepsilon}} [1 + o(1)].$$

We compute:

$$\gamma_1 = \alpha_0 \beta_1 + \sum_{n \neq 1} \alpha_n \beta_{1-n}$$

We start with computing $\alpha_0 \beta_1$. To do that, we deduce from equation (8.23) that:

$$\int_{\varphi_r^+}^0 \kappa_i(\varphi) d\varphi + \int_{\varphi_r^-}^0 \kappa_i(\varphi) d\varphi - \int_{\sigma_-} (\kappa_i(\varphi) - \pi) d\varphi = \Phi_d(E) + iS(E).$$

$$\alpha_0 \beta_1 = ie^{-\frac{i}{\varepsilon} (\int_{\varphi_r^+}^0 \kappa_i + \int_{\varphi_r^-}^0 \kappa_i)} e^{\frac{2i}{\varepsilon} \int_0^{\varphi_r^+} (\kappa_i - \pi)} e^{\int_0^{\varphi_r^+} \omega_+^i + \int_{\varphi_r^-}^0 \omega_-^i - \int_{\sigma_+} (\omega_+^i - \omega_-^i)} [1 + o(1)] + O(e^{\frac{-\alpha}{\varepsilon}}) e^{-S(E)/\varepsilon}.$$

Equation (8.23) leads to:

$$\int_{\varphi_r^+}^0 \kappa_i(\varphi) d\varphi + \int_{\varphi_r^-}^0 \kappa_i(\varphi) d\varphi - \int_{\sigma_-} (\kappa_i(\varphi) - \pi) d\varphi = \Phi_d(E) + iS(E).$$

Besides, according to Lemma 8.3, we have:

$$\int_0^{\varphi_r^+} \omega_+^i + \int_{\varphi_r^-}^0 \omega_-^i - \int_{\sigma_+} (\omega_+^i - \omega_-^i) = i\rho.$$

and:

$$\alpha_0 \beta_1 = ie^{-S/\varepsilon} e^{-i\Phi_d/\varepsilon} e^{i\rho} [1 + o(1)].$$

Since $S(E) \leq 2\pi \operatorname{Im} \varphi_i(E)$, we estimate the remainder in the expansion:

$$|\sum_{n \neq 0} \alpha_n \beta_{-n}| = o(e^{-S/\varepsilon}).$$

Finally, for $p \neq 0, 1$, we estimate:

$$\gamma_p = \sum_{n \in \mathbb{Z}} \alpha_n \beta_{p-n}.$$

For $p > 1$, we have:

$$|\gamma_p| = e^{-S/\varepsilon} e^{-\alpha/\varepsilon} O(e^{-\frac{2\pi Y_0(p-1)}{\varepsilon}}).$$

Similarly, we estimate for $p < 0$,

$$|\gamma_p| = e^{-S/\varepsilon} e^{-\alpha/\varepsilon} O(e^{-\frac{2\pi Y_0(|p|-1)}{\varepsilon}}).$$

- Now, we consider $\varphi \in \mathbb{R}$. We compute the Fourier asymptotic expansion of the coefficient $d(E, \varphi, \varepsilon)$ in a neighborhood \mathcal{U}_0 of E_0 :

$$\begin{aligned} d(\varphi, E, \varepsilon) &= iw(f_i, \sigma_i(f_i)^*) \left(\lambda_0(E, \varepsilon) + \sum_{n \in \mathbb{N}^*} (\lambda_n(E, \varepsilon) e^{\frac{2in\pi\varphi}{\varepsilon}} + (\lambda_n)^*(E, \varepsilon) e^{\frac{-2in\pi\varphi}{\varepsilon}}) \right) \\ &= i(w_0 k'_i)(E - W(0)) \sum_{n \in \mathbb{N}} u_n(\varphi, E, \varepsilon). \end{aligned}$$

where $u_n(\varphi, E, \varepsilon) = \lambda_n(E, \varepsilon) e^{\frac{2in\pi\varphi}{\varepsilon}} + (\lambda_n)^*(E, \varepsilon) e^{\frac{-2in\pi\varphi}{\varepsilon}}$, pour $n \in \mathbb{N}^*$, et $u_0(\varphi, E, \varepsilon) = \lambda_0(E, \varepsilon)$. We have:

$$u_0(\varphi, E, \varepsilon) = \gamma_0(E, \varepsilon) - \gamma_0^*(E, \varepsilon) = -ie^{i\frac{\Phi}{\varepsilon}} g(E, \varepsilon) - ie^{-i\frac{\Phi}{\varepsilon}} g^*(E, \varepsilon).$$

where $g(E, \varepsilon) = 1 + o(1)$.

We define $g(E, \varepsilon) = r_g(E, \varepsilon) e^{i\theta_g(E, \varepsilon)}$ where the functions $E \mapsto r_g(E, \varepsilon)$ and $E \mapsto \theta_g(E, \varepsilon)$ are analytic and satisfy

$$r_g^* = r_g, \quad r_g = 1 + o(1) \quad \theta_g^* = \theta_g, \quad \theta_g = o(1).$$

We simplify:

$$u_0(\varphi, E, \varepsilon) = -ir_g(E, \varepsilon) \cos\left(\frac{\Phi(E)}{\varepsilon} + \theta_g(E, \varepsilon)\right).$$

Similarly, we compute:

$$u_1(\varphi, E, \varepsilon) = ir_h(E, \varepsilon) e^{-S(E)/\varepsilon} \cos\left(\frac{\Phi_d + 2\pi\varphi}{\varepsilon} + \rho + \theta_h(E, \varepsilon)\right).$$

where the functions $E \mapsto r_h(E, \varepsilon)$ and $E \mapsto \theta_h(E, \varepsilon)$ are analytic and satisfy

$$r_h^* = r_h, \quad r_h = 1 + o(1) \quad \theta_h^* = \theta_h, \quad \theta_h = o(1).$$

In addition, we have the following estimate of the remainder:

$$\left| \sum_{p \geq 2} u_p(\varphi, E, \varepsilon) \right| \leq C e^{\frac{-S(E)}{\varepsilon}} e^{\frac{-\alpha}{\varepsilon}} \quad \text{pour } \varphi \in \mathbb{R}.$$

- We have proved that, for E in a neighborhood of E_0 , the Fourier expansion of $d(E, \varphi, \varepsilon)$ can be written:

$$\frac{d(\varphi, E, \varepsilon)}{i(w_0 k'_i)(E - W(0))} = -i[1 + o(1)] \cos\left(\frac{\Phi(E)}{\varepsilon} + o(1)\right) \quad (8.30)$$

$$+ i[1 + o(1)] e^{\frac{-S(E)}{\varepsilon}} \cos\left(\frac{\Phi_d + 2\pi\varphi}{\varepsilon} + \rho + o(1)\right) + e^{\frac{-S(E)}{\varepsilon}} O(e^{\frac{-\alpha}{\varepsilon}}).$$

The compactness of J implies that there exists a finite number of intervals $\{J_k\}_{k \in \{1 \dots p\}}$ such that:

1. $J \subset \bigcup_{k \in \{1, \dots, p\}} J_k$
2. For any $k \in \{1, \dots, p-1\}$, the intervals J_k and J_{k+1} overlap.
3. For any $k \in \{1, \dots, p\}$, there exists a complex neighborhood \mathcal{U}_k of J_k such that the expansion (8.30) is satisfied on \mathcal{U}_k .

We shall prove that we can define some functions $\tilde{\Phi}$ and $\tilde{\Phi}_d$ on the whole neighborhood $\mathcal{V} = \bigcup_{k \in \{1, \dots, p\}} \mathcal{U}_k$. To do that, we shall “stick” the expansions obtained on each interval.

The coefficient u_0 is written:

$$\begin{aligned} \forall E \in J_k, \quad u_0(E, \varepsilon) &= r_{0,k}(E, \varepsilon) \cos \left(\frac{\Phi(E)}{\varepsilon} + \theta_{0,k}(E, \varepsilon) \right) \\ \forall E \in J_{k+1}, \quad u_0(E, \varepsilon) &= r_{0,k+1}(E, \varepsilon) \cos \left(\frac{\Phi(E)}{\varepsilon} + \theta_{0,k+1}(E, \varepsilon) \right) \end{aligned}$$

where $r_{0,k}(E, \varepsilon) = 1 + o(1)$ and $\theta_{0,k}(E, \varepsilon) = o(1)$ (resp. $r_{0,k+1}(E, \varepsilon) = 1 + o(1)$ and $\theta_{0,k+1}(E, \varepsilon) = o(1)$) for $E \in J_k$ (resp. $E \in J_{k+1}$). We get that:

$$r_{0,k}(E, \varepsilon) = r_{0,k+1}(E, \varepsilon) = r_0(E, \varepsilon) \text{ et } \theta_{0,k}(E, \varepsilon) = \theta_{0,k+1}(E, \varepsilon) = \theta_0(E, \varepsilon) \text{ for } E \in J_k \cap J_{k+1}$$

The function $\tilde{\Phi}$ defined by its restrictions to each \mathcal{U}_k is analytic on \mathcal{V} .

The case of Φ_d is treated similarly.

Defining

$$F(\varphi, E, \varepsilon) = \frac{d(\varphi, E, \varepsilon)}{i(w_0 k'_i)(E - W(0)) r_0(E, \varepsilon)},$$

we finish the proof of Proposition 10.

8.5 Localization of the eigenvalues

In this section, we deduce Theorem 1 from Proposition 10.

We solve equation $F(\varphi, E, \varepsilon) = 0$, where F is described in (8.29).

8.5.1 Energy levels $E^{(l)}(\varepsilon)$

For $E \in \mathcal{V}$, we start with solving:

$$\cos \frac{\tilde{\Phi}(E, \varepsilon)}{\varepsilon} = 0 \tag{8.31}$$

$E \mapsto \tilde{\Phi}(E, \varepsilon)$ is a real analytic function. For a sufficiently small ε_0 , by Lemma 2.2, there exists a constant $m > 0$ such that:

$$\forall E \in \mathcal{V}, \quad \forall \varepsilon \in]0, \varepsilon_0[, \quad |\tilde{\Phi}'(E, \varepsilon)| \geq m \tag{8.32}$$

Consequently, equation (8.31) has a finite number of zeros in \mathcal{J} . We denote them by $E^{(l)}(\varepsilon)$, for $l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon)\}$. They are given by:

$$\frac{\tilde{\Phi}(E^{(l)}(\varepsilon), \varepsilon)}{\varepsilon} = l\pi + \frac{\pi}{2}, \quad \forall l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon)\}. \tag{8.33}$$

and satisfy:

$$E^{(l+1)}(\varepsilon) - E^{(l)}(\varepsilon) = \frac{1}{\tilde{\Phi}'(E^{(l)}(\varepsilon))} \pi \varepsilon + o(\varepsilon). \tag{8.34}$$

The distances between two consecutive zeros are of order ε . Precisely, by combining (8.32) with (8.34), we obtain that there exists a constant $c > 0$ such that:

$$\frac{1}{c} \varepsilon < |E^{(l+1)}(\varepsilon) - E^{(l)}(\varepsilon)| < c \varepsilon, \quad \forall l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon) - 1\} \tag{8.35}$$

First, we prove that the zeros of F are in an exponentially small neighborhood of the points $E^{(l)}(\varepsilon)$.

8.5.2 First order approximation

We give a first order approximation of the zeros of F .

We set

$$a_0(E, \varepsilon) = \cos \frac{\tilde{\Phi}(E, \varepsilon)}{\varepsilon}.$$

We can assume that the neighborhood \mathcal{V} is sufficiently small and such that, for any $E \in \mathcal{V}$,

$$\operatorname{Re}(S(E)) > \beta > 0.$$

Then, there exists a positive constant A such that

$$|F(\varphi, E, \varepsilon) - a_0(E, \varepsilon)| < Ae^{-\beta/\varepsilon}.$$

In addition, we have the following inequality:

$$\exists C > 0 / \quad \left| \cos \frac{\tilde{\Phi}(E, \varepsilon)}{\varepsilon} \right| \geq \frac{C}{\varepsilon} d(E, \bigcup_{l \in \{L_-, \dots, L_+\}} E^{(l)}(\varepsilon)). \quad (8.36)$$

Actually, there exists a constant $c > 0$ such that:

$$|\cos \theta| \geq cd(\theta, \pi\mathbb{Z} + \pi/2).$$

By using (8.32), we obtain the relation:

$$|\tilde{\Phi}(E, \varepsilon) - \tilde{\Phi}(E^{(l)}(\varepsilon), \varepsilon)| \geq m|E - E^{(l)}(\varepsilon)|$$

and finally:

$$\left| \cos \frac{\tilde{\Phi}(E, \varepsilon)}{\varepsilon} \right| \geq \frac{C}{\varepsilon} d\left(E, \bigcup_{l \in \{L_-, \dots, L_+\}} E^{(l)}(\varepsilon)\right).$$

For $z_0 \in \mathbb{C}$ and $r > 0$, we define

$$D(z_0, r) = \{z \in \mathbb{C} ; |z - z_0| < r\}.$$

Inequality (8.36) implies that there are no zeros of F outside exponentially small neighborhoods of the points $E^{(l)}(\varepsilon)$. Precisely, there exists a positive constant D such that, if $r \geq D\varepsilon e^{-\beta/\varepsilon}$, then for any $E \in \partial D(E^{(l)}(\varepsilon), r)$, we have:

$$|F(\varphi, E, \varepsilon) - a_0(E, \varepsilon)| < |a_0(E, \varepsilon)|.$$

Rouché's Theorem implies that, for any l , F has exactly one zero $E_l(\varphi, \varepsilon)$, in each neighborhood $D(E^{(l)}(\varepsilon), D\varepsilon e^{-\beta/\varepsilon})$ of $E^{(l)}(\varepsilon)$. The relation $F = F^*$ allows us to recover that the eigenvalues are real. Indeed, if $F(E) = 0$, \overline{E} is also a zero of F . By uniqueness, we obtain that $E = \overline{E}$.

We set:

$$E_l(\varphi, \varepsilon) = E^{(l)}(\varepsilon) + \varepsilon\lambda_l(\varphi, \varepsilon).$$

We know that $\lambda_l(\varphi, \varepsilon)$ is exponentially small. Now, we compute its asymptotic behavior.

8.5.3 Second order approximation

We define:

$$a_1(\varphi, E, \varepsilon) = F(\varphi, E, \varepsilon) - a_0(E, \varepsilon).$$

We write

$$e^{-S(E_l(\varphi, \varepsilon))/\varepsilon} = e^{-S(E^{(l)}(\varepsilon))/\varepsilon}(1 + O(\lambda_l(\varphi, \varepsilon))).$$

Similarly, with the help of the modified phase $\widetilde{\Phi}_d$, we obtain the expansion:

$$\cos\left(\frac{\widetilde{\Phi}_d(E_l(\varphi, \varepsilon)) + 2\pi\varphi + \rho\varepsilon}{\varepsilon}\right) = \cos\left(\frac{\widetilde{\Phi}_d(E^{(l)}(\varepsilon)) + 2\pi\varphi + \rho\varepsilon}{\varepsilon}\right) + O(e^{-\beta/\varepsilon})$$

The expansion of a_1 can be written:

$$a_1(\varphi, E_l(\varphi, \varepsilon), \varepsilon) = a_1(\varphi, E^{(l)}(\varepsilon), \varepsilon)(1 + r(\varphi, E^{(l)}(\varepsilon), \varepsilon)).$$

Moreover, we use the first order Taylor's expansion of the function $E \mapsto a_0(E, \varepsilon)$:

$$a_0(E_l(\varphi, \varepsilon), \varepsilon) = (-1)^{l+1}\widetilde{\Phi}'(E^{(l)}(\varepsilon), \varepsilon)\lambda_l(\varphi, \varepsilon)(1 + r(\varphi, E^{(l)}, \varepsilon)) = (-1)^{l+1}\Phi'(E^{(l)})\lambda_l(\varphi, \varepsilon)(1 + o(1)).$$

By combining these computations, we finally obtain:

$$\lambda_l(\varphi, \varepsilon) = \frac{(-1)^{l+1}}{\Phi'(E^{(l)}(\varepsilon))}e^{-S(E^{(l)}(\varepsilon))/\varepsilon}\left(\cos\left(\frac{\widetilde{\Phi}_d(E^{(l)}(\varepsilon), \varepsilon) + 2\pi\varphi}{\varepsilon} + \rho\right) + o(1)\right).$$

8.6 Application to the trace formula

In [4], the author proves the existence of an asymptotic expansion of $\text{tr}[f(H_{\varphi, \varepsilon})]$, for $f \in C_0^\infty$, when $\text{Supp } f$ is disjoint from the bands of H_0 ; in addition, he computes explicitly the first and second terms of this expansion.

Corollary 1 allows us to recover these terms.

8.6.1

Let J be an interval satisfying (H_J) . Particularly, J is such that $J \cap (\sigma_{ac} \cup \sigma_{sc}) = \emptyset$. For $f \in C_0^\infty$, with $\text{Supp } f \subset J$, we compute:

$$\text{tr}[f(H_{\varphi, \varepsilon})] = \sum_{l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon)\}} f(E_l(\varphi, \varepsilon)).$$

Let $\beta > 0$ be such that $S(E) > \beta$ for any $E \in J$; according to Theorem 1, we know that there exists a constant $C > 0$ such that:

$$\forall u \in [0, \varepsilon], \quad |\text{tr}[f(H_{\varphi, \varepsilon})] - \text{tr}[f(H_{u, \varepsilon})]| < C \sum_{l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon)\}} \varepsilon e^{-\beta/\varepsilon}.$$

By integrating with respect to u , we obtain that:

$$\text{tr}[f(H_{\varphi, \varepsilon})] = \frac{1}{\varepsilon} \int_0^\varepsilon \text{tr}[f(H_{u, \varepsilon})] du + O(e^{-\beta/\varepsilon}).$$

According to Theorem 1, we know that there exists a constant C such that

$$\forall u \in [0, \varepsilon], \quad \left| \text{tr}[f(H_{u, \varepsilon})] - \sum_{l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon)\}} f(E^{(l)}(\varepsilon)) \right| < C e^{-\beta/\varepsilon}$$

By integration, we obtain:

$$\frac{1}{\varepsilon} \int_0^\varepsilon \text{tr}[f(H_{u, \varepsilon})] du = \sum_{l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon)\}} f(E^{(l)}(\varepsilon)) + O(e^{-\beta/\varepsilon}) \quad (8.37)$$

Now, we estimate:

$$\sum_{l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon)\}} f(E^{(l)}(\varepsilon)) = \sum_{l \in \{L_-(\varepsilon), \dots, L_+(\varepsilon)\}} f \circ \widetilde{\Phi}^{-1}(\varepsilon(l\pi + \pi/2))$$

8.6.2

Now, we compute this last term.

Lemma 8.4. *Let f be a function in C_0^∞ such that $\text{Supp } f \subset J$. The trace of $H_{\varphi, \varepsilon}$ has the following asymptotic behavior:*

$$\int_0^\varepsilon \text{tr}[f(H_{u, \varepsilon})]du = \frac{1}{\pi} \int_J f(\mathcal{E}) \tilde{\Phi}'(\mathcal{E}, \varepsilon) d\mathcal{E} + O(\varepsilon^\infty)$$

Proof The proof of this Lemma is based on elementary results of real analysis.

- We apply the Poisson formula to the function $f \circ \tilde{\Phi}^{-1} \in C_0^\infty$:

$$\varepsilon \sum_{l \in \mathbb{Z}} f \circ \tilde{\Phi}^{-1}(\varepsilon(l\pi + \pi/2)) = 2 \sum_{n \in \mathbb{Z}} (-1)^n (\widehat{f \circ \tilde{\Phi}^{-1}}) \left(\frac{2n}{\varepsilon} \right).$$

Besides, the Fourier transform of $f \circ \tilde{\Phi}^{-1}$ satisfies the estimates:

$$\forall \nu > 1, \quad \exists C_\nu > 0, \text{ such that } \left| (\widehat{f \circ \tilde{\Phi}^{-1}}) \left(\frac{2n}{\varepsilon} \right) \right| \leq C_\nu \frac{\varepsilon^\nu}{n^\nu}.$$

Actually, since $f \circ \tilde{\Phi}^{-1}$ is C^ν , $|\xi^\nu f \circ \tilde{\Phi}^{-1}(\xi)|$ is bounded.

This leads to:

$$\varepsilon \sum_{p \in \mathbb{Z}} f \circ \tilde{\Phi}^{-1}(\varepsilon(p\pi + \pi/2)) = 2(\widehat{f \circ \tilde{\Phi}^{-1}})(0) + O(\varepsilon^\infty).$$

- It remains to prove that:

$$2(\widehat{f \circ \tilde{\Phi}^{-1}})(0) = \frac{1}{\pi} \int_J f \circ \tilde{\Phi}^{-1}(u) du.$$

With the substitution $u = \tilde{\Phi}(\mathcal{E})$, we obtain that:

$$2(\widehat{f \circ \tilde{\Phi}^{-1}})(0) = \frac{1}{\pi} \int_J f(\mathcal{E}) \tilde{\Phi}'(\mathcal{E}, \varepsilon) d\mathcal{E}$$

This completes the proof of Lemma 8.4.

◇

8.6.3 Conclusion

To get an asymptotic expansion of the trace at any order, it suffices to know an asymptotic expansion of the modified phase at any order. Our computations are not accurate enough, but we know that $\tilde{\Phi}'(\mathcal{E}) = \Phi'(E) + o(\varepsilon)$, hence:

$$\frac{1}{\pi} \int_J f(\mathcal{E}) \tilde{\Phi}'(\mathcal{E}) d\mathcal{E} = \frac{1}{\pi} \int_J f(\mathcal{E}) \Phi'(\mathcal{E}) d\mathcal{E} + o(\varepsilon).$$

To transform the right member of previous equality, we do the substitution $(\kappa, u) \mapsto (E(\kappa) + W(u), u)$, which implies:

$$\frac{1}{\pi} \int_J f(\mathcal{E}) \Phi'(\mathcal{E}) d\mathcal{E} = \frac{1}{2\pi} \int_{[-\pi, \pi]} \int_{\varphi_r^-}^{\varphi_r^+} f(E(\kappa) + W(u)) d\kappa du$$

We finally obtain:

$$\int_0^\varepsilon \text{tr}[f(H_{u, \varepsilon})] du = \frac{1}{2\pi} \int_{[-\pi, \pi]} \int_{\varphi_r^-}^{\varphi_r^+} f(E(\kappa) + W(u)) d\kappa du + o(\varepsilon)$$

This ends the proof of Corollary 1.

8.7 Asymptotic behavior of the eigenvalues

Now, we give a second application of Theorem 1 for the computation of the asymptotic behavior of the eigenvalues of $H_{\varphi, \varepsilon}$. Such a computation is outlined in [3], in the case $V = 0$. We obtain an explicit result at first order.

Under the assumptions of Theorem 1, E_r is the only end of $\sigma(H_0)$ belonging to $(E - W)(\mathbb{R})$. We define:

$$d_p(E_n) = \lim_{\substack{E \rightarrow E_n, \\ E \in [E_n, E_p]}} \frac{k(E) - k(E_n)}{\sqrt{E - E_n}} \quad (8.38)$$

Corollary 2. *Let $H_{\varphi, \varepsilon}$ verify the assumptions of Theorem 1. The eigenvalues $E^{(l)}(\varphi, \varepsilon)$ of $H_{\varphi, \varepsilon}$ have the following asymptotic behavior:*

$$E^{(l)}(\varphi, \varepsilon) = \tilde{\Phi}^{-1}(\varepsilon(l\pi + \pi/2)) + o(\varepsilon^\infty).$$

Particularly, $E^{(l)}(\varphi, \varepsilon)$ has the following Taylor expansion at first order in ε :

$$E^{(l)}(\varphi, \varepsilon) = E_r + W(0) + \sqrt{\frac{W''(0)}{2}} \frac{1}{d_i(E_r)} (2l + 1)\varepsilon + o(\varepsilon),$$

where $d_i(E_r)$ is defined by (8.38).

Proof The first equality is obvious. It suffices to give an expansion of $\tilde{\Phi}^{-1}(\varepsilon(l\pi + \pi/2))$. To do that, we compute an expansion at first order of:

$$\Phi(E_r + W(0) + \alpha) = \int_{\varphi_r^-(E_r + W_- + \alpha)}^{\varphi_r^+(E_r + W_- + \alpha)} k(E_r + W(0) + \alpha - W(u)) du.$$

The mapping W is a bijection from $[0, \varphi_r^+]$ to $[W(0), E_r]$. By the substitution $\alpha v = W(0) + \alpha - W(u)$, we get that:

$$\int_0^{\varphi_r^+(E_r + W(0) + \alpha)} = \alpha \int_0^1 \frac{k(E_r + \alpha v)}{W' \circ W^{-1}(W(0) + \alpha(1-v))} dv.$$

But, $\lim_{\alpha \rightarrow 0} \frac{k(E_r + \alpha v)}{W' \circ W^{-1}(W_- + \alpha(1-v))} = \frac{d_i(E_r)}{\sqrt{2W''(0)}} \frac{\sqrt{v}}{\sqrt{1-v}}$.

Similarly, on $[\varphi_r^-, 0]$, we have:

$$\Phi(E_r + W(0) + \alpha) = d_i(E_r) \frac{\pi}{2} \sqrt{\frac{2}{W''(0)}} \alpha [1 + o(1)].$$

Consequently, by inverting the expansion of $\tilde{\Phi}$ in the neighborhood of $E_r + W(0)$, we prove the result. \diamond
We point out that, as in 8.6, a more accurate asymptotic expansion of $\tilde{\Phi}$ would give a better result on the eigenvalues.

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